14.5 The Chain Rule

Case 0: CR in Case I

Given:

\[ x = g(t) \quad \text{and} \quad y = f(x) \]

\[ \text{Slope} = f'(x_0) \]

\[ \text{Slope} = g'(t_0) \]

\[ \text{Slope} = (f \circ g)'(t_0) \]

Form Composition

\[ \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \]

Chain Rule

- Slope of L3 = Slope of L2 \times \text{Slope of } L_1

\[ (f \circ g)'(t_0) = f'(g(t_0)) \cdot g'(t_0) \]
CASE I

OR FOR FUNCTIONS ON CURVES

\( (x, y) = \mathbf{r}(t) = (\cos t, \sin t) \)

where \( \mathbf{r}: \mathbb{R} \rightarrow \mathbb{R}^2 \) is curve in plane.

\[ z = f(x, y) = 3x^2 + 4y^2 = \text{TEMPERATURE} \]

AT PT \((x, y)\) IN PLANE

How does temperature of at change with time \( \frac{dz}{dt} \) at \( t = \pi/4 \)?

FIND \( \frac{dz}{dt} \)

METHOD 1 From the Composition,

\[ z(t) = f(x(t), y(t)) = f(x(t), y(t)) \]

\[ = 3\cos^2 t + 4\sin^2 t \]

\[ = 3 + 2\cos^2 t \]

\[ \Rightarrow \frac{dz}{dt} = 2\sin t \cos t \]

\( \frac{dz}{dt}(\pi/4) = 2 \frac{1}{\sqrt{2}} \frac{\sqrt{2}}{2} = 1 \).
METHOD II: The CALE III Chain Rule

\[
\frac{dz}{dt} = \frac{dz}{dx} \frac{dx}{dt} + \frac{dz}{dy} \frac{dy}{dt}
\]

\[
\frac{dx}{dt} = \frac{dy}{dt} + \frac{dy}{dt}
\]

**EX**

\[
\frac{dz}{dx} = 6x \quad \frac{dz}{dy} = 8y
\]

\[
\frac{dx}{dt} = -8\sin t \quad \frac{dy}{dt} = \cos t
\]

So,

\[
\frac{dz}{dt} = 6x(t) \cdot (-8\sin t) + 8y(t) \cdot \cos t
\]

\[
= -48 \cos t \sin t + 8 \cos t \sin t
\]

\[
= 2 \cos t \sin t
\]

**Note** Just like in CALE I, must evaluate derivatives of outer function (\( z = f(x, y) \)) at the values of inner functions (\( x = x(t) \) and \( y = y(t) \)).
Suppose \( z = f(x, y) \) \( (x, y) = t(t) \)

\( t(0) = (1, 3) \)
\( t'(0) = (2, 5) \)

\( f(1, 3) = 4 \)
\( \frac{df}{dx}(1, 3) = 6 \)
\( \frac{df}{dy}(1, 3) = 7 \)

Then

\[
\frac{dz}{dt}(0) = \frac{df}{dx}(t(0)) \frac{dx}{dt}(0) + \frac{df}{dy}(t(0)) \frac{dy}{dt}(0)
\]

\[
= \frac{df}{dx}(1, 3) \frac{dx}{dt}(0) + \frac{df}{dy}(1, 3) \frac{dy}{dt}(0)
\]

\[
= 6 \times 2 + 7 \times 5 = 47
\]

Alternate form of CR for functions on curves

\( \pi : \mathbb{R}^1 \to \mathbb{R}^2 \)
\( f : \mathbb{R}^2 \to \mathbb{R}^1 \)

\( \pi(t) = (x(t), y(t)) \)
\( z = f(x(t), y(t)) \)

\( \pi'(t) = (x'(t), y'(t)) \)

\( \nabla f = \left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right] \)

"Gradient up \( f \)"
\[ f_0 : \mathbb{R} \rightarrow \mathbb{R} \]

\[ (f_0 \circ \vec{r})' (t) = \frac{\partial f_0}{\partial x} (\vec{r} (t)) \frac{dx}{dt} (t) + \frac{\partial f_0}{\partial y} (\vec{r} (t)) \frac{dy}{dt} (t) \]

\[ = \left( \frac{\partial f_0}{\partial x} (\vec{r} (t)), \frac{\partial f_0}{\partial y} (\vec{r} (t)) \right) \cdot \left( \frac{dx}{dt} (t), \frac{dy}{dt} (t) \right) \]

\[ = \nabla f_0 (\vec{r} (t)) \cdot \vec{r}' (t) \]

**Case 2**

**Chain Rule for Functions on Surfaces**

- \( \vec{r} : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \)
- **Parameterization of Surface**

\[ \vec{r} (u, v) = (u, v) \]

\[ x = u, \quad y = v, \quad z = u^2 + v^2 \]

**Double Cone**

\[ \nabla \cdot \mathbf{F} = 2 \]

**Function**

\[ w = f(x, y, z) = \frac{1}{3} x^2 + 4y^2 + 5z^2 = \text{Air Pressure at } (x, y, z) \]
Find \( \frac{\partial w}{\partial u} \) \( \text{at} \quad (u_0, v_0) = (0, 2) \)

\[ w(u, v) = f \left( \vec{r}(u, v) \right) \]
\[ = \text{Air Pressure at point on curve with parameter} \ (u, v) \]

\[ \vec{r} (0, 2) = (2, 0, 2) \]

So

\[ \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u} \]

\[ \frac{\partial w}{\partial u} (0, 2) = \frac{\partial w}{\partial x} (2, 0, 2) \frac{\partial x}{\partial u} (0, 2) + \frac{\partial w}{\partial y} (2, 0, 2) \frac{\partial y}{\partial u} (0, 2) \]
\[ + \frac{\partial w}{\partial z} (2, 0, 2) \frac{\partial z}{\partial u} (0, 2) \]

\[ = (6x) \left( -\kappa_2 u \right) + (8y) \left( \kappa_2 v \right) + 107 \cdot (0) \]

\[ = 12 \cdot (-2, 0, 0) + 0 + 0 \]

\[ = (2, 0, 0) \]

As go around circle \( \frac{\partial w}{\partial u} = 2 \),
Ref. of Pressure \( 0 \) at \( (0, 2) \)