14.6 DIRECTIONAL DERIVATIVES AND THE GRADIENT

Given \( z = f(x, y) \) and a point \((x_0, y_0)\)

Recall \( \frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0)}{h} \)

Recast using vectors:
\[
\overrightarrow{\Delta} = (x, y), \quad \overrightarrow{x_0} = (x_0, y_0)
\]
\[
(x_0+h, y_0) = (x_0, y_0) + h \overrightarrow{\Delta} = \overrightarrow{x_0} + h \overrightarrow{\Delta}
\]

So \( \frac{\partial f}{\partial x}(\overrightarrow{x_0}) = \lim_{h \to 0} \frac{f(x_0+h \overrightarrow{\Delta}) - f(x_0)}{h} \)

\( \nabla \overrightarrow{x_0} = \text{RofC of } f \text{ at } \overrightarrow{x_0} \) in direction \( \overrightarrow{\Delta} \).

There is nothing special about the direction of \( \overrightarrow{\Delta} \),
we could go in \( \text{any } \) direction \( \overrightarrow{\Delta} \) from \( \overrightarrow{x_0} \).

Choose \( \| \overrightarrow{\Delta} \| = 1 \) always.
The directional derivative of \( f \) at \( \vec{x}_0 \) in direction \( \vec{u} \) is

\[
D_{\vec{u}} f \left( \vec{x}_0 \right) := \lim_{h \to 0} \frac{f(\vec{x}_0 + h\vec{u}) - f(\vec{x}_0)}{h}
\]

\# How to compute \( D_{\vec{u}} f \left( \vec{x}_0 \right) \)?

\# Use the gradient of \( f \).

The gradient of \( z = f(x, y) \) is the vector field

\[
\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = \left( \frac{\partial f}{\partial x} \right) \vec{i} + \left( \frac{\partial f}{\partial y} \right) \vec{j}
\]

\[\]

Example:

\[ z = f(x, y) = 3x^2 + 4xy + 5y^2 \]

Find \( \nabla f \):

\[ \nabla f(x, y) = (6x + 4y, 4x + 10y) \]

- gives a different vector at each \((x, y)\)
- \( \nabla f \) is a "vector field"

If plug in \((3, 2)\) get

\[ \nabla f (3, 2) = (6 + 8, 4 + 20) = (14, 24) \]

- A vector at \((3, 2)\)
\[
\text{THM: } (D_{\vec{u}} f)(\vec{x}_0) = \nabla f(\vec{x}_0) \cdot \vec{u}
\]

[Scalar \ Vector \ Vector ]

Example:
\[ z = f(x, y) = 3x^2 + 4xy + 5y^2. \]

\[ \vec{x}_0 = (1, 2), \quad \vec{u} = \frac{(2, 3)}{|(2, 3)|} = \frac{1}{\sqrt{13}}(2, 3) \]

Then
\[ (D_{\vec{u}} f)(\vec{x}_0) = \nabla f(\vec{x}_0) \cdot \vec{u} \]

\[ = \nabla f(1, 2) \cdot \frac{1}{\sqrt{13}}(2, 3) \]

Above:
\[ = \frac{1}{\sqrt{13}}(14, 24) \cdot (2, 3) = \sqrt{13}(100). \]

(1) Find Dir. Der. of \( f \) at \( \vec{x}_0 = (1, 2) \) in \( \theta = \frac{\pi}{4} \).

\[ \text{Use } \vec{u} = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \text{ unit length.} \]

\[ D_{\vec{u}} f(\vec{x}_0) = (14, 24) \cdot \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = \frac{1}{2} \cdot 38. \]
PROOF OF THM

We have \( Z = f(x, y) \) @ \( \vec{z}_0 = (x_0, y_0) \)

Let \( C \) be curve obtained by slicing

Surface \( Z = f(x, y) \) in the vertical plane

Through \( \vec{z}_0 \) containing vectors \( \vec{u} \) and \( \vec{t} \).

\[
\begin{align*}
\text{Plane slope} & = (D_u f)(\vec{z}_0) \\
\text{Curve} C & = (f \circ \vec{r})(t)
\end{align*}
\]

Let \( \vec{r}(t) = \vec{z}_0 + t \vec{v} \) parametrize line in \( z \) \( x \) plane

Then \( \vec{z}_0 \) in direction \( \vec{u} \)

So \( \vec{r}(0) = \vec{z}_0 \), \( \vec{r}'(0) = \vec{u} \).

Let \( g(t) = f(\vec{r}(t)) = (f \circ \vec{r})(t) \) be

Restriction of \( f \) to this line.

Claim \( (D_u f)(\vec{z}_0) = \text{RIF} \) of \( g \) at \( t=0 \)
\[
(D_{\mathbf{u}} f)(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}
\]

\[
= \lim_{h \to 0} \frac{f(t(u)) - f(r(0))}{h}
\]

\[
= \lim_{h \to 0} \frac{g(h) - g(0)}{h}
\]

\[
= g'(0)
\]

\[
= (g \circ r)'(0)
\]

\[
= D_f r(0) \cdot \mathbf{u}
\]

Cool!
DIRECTION OF STEEPEST ASCENT

Think surface \( S = \text{Graph of } z = f(x, y) \).

You are at \( \vec{r}_0 \) in xy-plane and walk in direction \( \vec{u} \).

Your friend walks on \( S \) immediately above you.

What dir. \( \vec{u} \) should you walk so that your friend goes up hill, i.e., steepest ascent?

How do we choose \( \vec{u} \) to maximize \( \nabla f(\vec{r}_0) \)?

\[
(\nabla f(\vec{r}_0)) = \nabla f(\vec{r}_0) \cdot \vec{u} = |\nabla f(\vec{r}_0)| |\vec{u}| \cos \theta = |\nabla f(\vec{r}_0)| \cos \theta \quad \text{as } |\vec{u}| = 1
\]

So biggest when \( \theta = 0 \).

So want \( \vec{u} \parallel \nabla f(\vec{r}_0) \).
SUMMARY

Physics Interpretation of Vector $\nabla f(\mathbf{\xi})$

Direction of Steepest Ascent $\mathbf{u}$

$$\mathbf{u} = \frac{\nabla f(\mathbf{\xi})}{|\nabla f(\mathbf{\xi})|}$$

$\nabla f$ of $f$ in that $\text{dir}^n = \frac{\nabla f(\mathbf{\xi})}{|\nabla f(\mathbf{\xi})|}$

Example $z = f(\mathbf{x}y) = 3x^2 + 4xy + 5y^2$ @ $\mathbf{\xi}_0 = (1, 2)$

$\nabla f(1, 2) = (14, 24)$

So

$$\mathbf{u} = \frac{1}{\sqrt{14^2 + 24^2}} (14, 24) = \text{Dir}^n \text{ of Steepest Ascent}$$

$|\nabla f(1, 2)| = \sqrt{14^2 + 24^2} = \text{Max} \text{ Rate of } f @ (1, 2)$

Similarly

$\text{Dir}^n \text{ of Steepest Descent} = \mathbf{u} = \frac{-\nabla f(\mathbf{\xi})}{|\nabla f(\mathbf{\xi})|}$

with

$$\nabla f$$

$(\mathbf{\xi})$
II. GRADIENTS + LEVEL CURVES

Let \( z = f(x, y) \) at \( \vec{x}_0 \).

Direction of steepest ascent \( \perp \) level curve through \( \vec{x}_0 \).

Let \( \vec{v}(t) \) parametrize the level curve of \( f \) through \( \vec{x}_0 \).

Choose \( \vec{v}(0) = \vec{x}_0 \).

We know \( (f \circ \vec{v})(t) = k \).

So \( \vec{0} = \left( f \circ \vec{v} \right)'(0) \).

So \( \nabla f(\vec{x}_0) \perp \vec{v}'(0) = \nabla f(\vec{x}_0) \cdot \vec{v} \).
Ex: Parametrize tangent line to curve 
\[ x^2 + y^3 = 5 \quad \text{in } \mathbb{R}^2 \]

@ \((x_0, y_0) = (2, 1) = \vec{p}\)

Well 
\[ \vec{r}(t) = \vec{p} + t\vec{v} \quad \vec{v} \perp \nabla f(\vec{p}) \]

\[ f(x, y) = x^2 + y^3 \]

\[ \nabla f = (2x, 3y^2) = (4, 3) \quad \vec{v} \]

So choose 
\[ \vec{v} = (3, -4) \]

\[ \vec{r}(t) = (2, 1) + t(3, -4) \]

III: Normal to Level Surface

If \( S \) is Level Surface of \( F \),
\[ F(x, y, z) = c \quad \text{on } S \]

and \((x_0, y_0, z_0) \in S\) Then 
\[ \vec{n} \]

Normal to \( S \) at \((x_0, y_0, z_0) = \vec{n} = \frac{\nabla f(x_0, y_0, z_0)}{|| \nabla f(x_0, y_0, z_0) ||} \]
Suppose $S$ is a level surface

$$F(x, y, z) = x^2 + y^2 + z^2 = 14.$$ 

$\vec{p} = (1, 2, 3) \in S$.

Find unit normal at $\vec{p}$.

Well

$$\nabla F = (2x, 2y, 2z) = (2, 4, 6) \vec{n}$$

So

$$\vec{n} = \frac{1}{\sqrt{4 + 16 + 36}} (2, 4, 6)$$