14.8 CONstrained Optimization + Method of

LAGRANGIAN MULTIPLIERS

Most real-world opt^n problems are constrained.

SIMPLE CASE (only one b = wo)

Find Hess MAX+MIN of

\[ \mathcal{z} = f(x,y) \]

OBJECTIVE FUNCTION

subject to constraint that \((x,y)\) lie on

the curve \(C\) in plane given as a level curve of a 2nd \(f^a\),

\[ \mathcal{g}(x,y) = k \]

CONSTRAINT EQU.

THINK

\[ \mathcal{z} = f(x,y) = \text{ELEVATION @ Lake Tahoe} \]

\[ \mathcal{g}(x,y) = k \]

is a SNOW-SHOE TRAIL

Find highest + lowest elevation on the trail
Quick Theory

Let \((y = \bar{v}(t))\) parameterize constraint curve \(C\) (trail)

Then \(h(t) = f(\bar{v}(t))\) is height along trail

Find \(\text{absolute max/min of } h\).

Method I (See 14.7B EX 2)

1. Calculate formula for \(h\)
2. Solve \(\text{case I } \Rightarrow \text{max/min problem}\)

Method II (Lagrange Multipliers)

Critical points of \(h\) satisfy

\[ 0 = h'(t) = \nabla f(\bar{v}(t)) \cdot \bar{v}'(t) \]

\(\Rightarrow\) must be normal to \(C\) at CPT.

But by 14.6 we know \(\nabla g\) is always normal to level curve \(g(x, y) = k\).

So at CPT \(\nabla f \parallel \nabla g\)

Or level curve of \(f\) tangent to level curve of \(g\).
Find max/min of \( z = f(x,y) = y^2 - x^2 \) on curve \( g(x,y) = x^2 + y^2 = 1 \).

**Strategy**

1. Sketch level curves \( f(x,y) = c \) of \( f \).
2. What is max value of \( c \) for which level curve of \( f \) intersects constraint curve \( x^2 + y^2 = 1 \).

(i.e. What is highest elevation \( c \), along show that \( f \) and \( x^2 + y^2 = 1 \) have a common tangent.)
As $c \uparrow$, last $c$-value for which level curve intersect is one for which target line to $g=k$ agrees with target line to $f=c$.

Once Again: $\nabla f \parallel \nabla g$.

Or $\nabla f = \lambda \nabla g$ for some $\lambda$.

$\lambda = \text{Lagrangian multiplier } \ (\lambda = L)$

**Method:** Find value of $f$ at all points $(x_0, y_0)$ for which there is a $\lambda$ so that

1. $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$

And 2. $g(x_0, y_0) = k \quad \text{(must be an constant)}$. 
Example continued

\[ \nabla g = (2x, 2y) \]
\[ \nabla f = (-2x, 2y) \]

Get \((2x, 2y) = \lambda (2x, 2y)\)

or

\[
\begin{align*}
-x &= \lambda x \quad (1) \\
y &= \lambda y \quad (2) \\
x^2 + y^2 &= 1 \quad (3)
\end{align*}
\]

3 M E Q U A T I O N S \n3 U N K N O W N S \((x, y, \lambda)\)

\[ \begin{align*}
(1+\lambda)x &= 0 \\
(1-\lambda)y &= 0 \\
x^2 + y^2 &= 1
\end{align*} \]

GET R I T S = 0 AND FACTOR

1. \(\lambda = -1\) or \(x = 0\)
2. \(\lambda = 1\)
3. \(-2y = 0 \Rightarrow y = 0\)

By 3, \(x = \pm 1\)

Get \((x, y, \lambda) = (\pm 1, 0, -1)\) \(\Rightarrow \frac{f(\pm 1, 0)}{\text{min}} = -1\)

or

\(x = 0\)

By 3, \(y = \pm 1\)

By 2, \(\lambda = 1\)

Get \((x, y, \lambda) = (0, \pm 1, 1)\) \(\Rightarrow f(0, \pm 1) = 1\) \(\text{max}\)
TREE DIAGRAM

TRACE ALL Possible Branches

\[ \lambda = 1 \]
\[ \gamma = 0 \]

\[ \gamma = -1 \]
\[ \gamma = +1 \]

\[ x = -1 \]
\[ x = +1 \]
\[ y = 1 \]
\[ y = -1 \]

EX. 2

Find \( \text{MAX} + \text{MIN} \) of \( f(x,y) = x^2 + y^2 \)
on rotated ellipse \( 4(x+y)^2 + (x-y)^2 = 1 \)

GEOMETRIC METHOD

LEVEL CURVES of \( f \)

CONSTRAINT ELLIPSE
EX2  Algebraic Method

Find maximum of \( z = f(x,y) = x^2 + y^2 \)
on \( g(x,y) = 4(x+y)^2 + 6(x-y)^2 = 1 \)
\[= 5x^2 + 5y^2 + 6xy = 1 \]

\[ f_x = \lambda g_x : \quad 2x = \lambda (10x + 6y) \quad \text{(1)} \]
\[ f_y = \lambda g_y : \quad 2y = \lambda (10y + 6x) \quad \text{(2)} \]
\[ g = 1 \quad 5x^2 + 5y^2 + 6xy = 1 \quad \text{(3)} \]

\( \text{(1)} - \text{(2)} : \quad 2(x-y) = \lambda [10(x-y) + 6(y-x)] \]

or \( (x-y)[10\lambda - 6\lambda - 2] = 0 \)

\( (x-y)(4\lambda - 2) = 0 \quad \text{(4)} \)

By \( \text{(4)} \) \( y = x \quad \text{or} \quad \lambda = \frac{1}{2} \)

\[ y = x \quad \text{By \( \text{(3)} \)} \quad 16x^2 = 1 \Rightarrow x = \pm \frac{1}{4} \]

So get \( (x,y) = (\pm \frac{1}{4}, \pm \frac{1}{4}) \)

By \( \text{(4)} \) \( \pm \frac{1}{2} = \lambda \left( \frac{10}{4} + \frac{6}{4} \right) \Rightarrow \lambda = \frac{1}{8} \).

\( \text{Ver} \ 2 \) hands with \( \text{Tese}(x,y,\lambda) \)

\( \lambda = \frac{1}{2} \quad \text{By \( \text{(4)} \)} \quad 4x = 10x + 6y \Rightarrow y = -x \)

\[ \text{By \( \text{(3)} \)} \quad 4x^2 = 1 \Rightarrow x = \pm \frac{1}{2} \]
So we get:

\[(x, y, \lambda) = \left\{ \begin{array}{c}
\left( \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right) \\
\left( -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)
\end{array} \right. \]

MAX

\[P = \frac{1}{2}\]

CHECK 2 HOURS.

TREE DIAGRAM

\[
\begin{array}{c}
\text{3} \\
\text{4} \\
\text{2}
\end{array}
\]

\[
\begin{array}{c}
\text{1} \\
\text{3}
\end{array}
\]

\[
\begin{array}{c}
\text{1} \\
\text{2}
\end{array}
\]

\[
\begin{array}{c}
\text{1} \\
\text{2}
\end{array}
\]

\[
\begin{array}{c}
\text{1} \\
\text{2}
\end{array}
\]

\[
\begin{array}{c}
\text{1} \\
\text{2}
\end{array}
\]

\[
\begin{array}{c}
\text{1} \\
\text{2}
\end{array}
\]
**ALGEBRAIC METHOD**

\[ g(x,y) = 4(x+y)^2 + (x-y)^2 \]
\[ = 5x^2 + 5y^2 + 6xy \]

\[ f_x = \lambda g_x; \quad 2x = \lambda (10x + 6y) \quad 1 \]
\[ f_y = \lambda g_y; \quad 2y = \lambda (10y + 6x) \quad 2 \]

\[ 1: \quad (2-10\lambda)x = 6\lambda y \quad 3 \]
\[ 2: \quad (2-10\lambda)y = 6\lambda x \quad 4 \]

\[ 3 \times 4: \quad (2-10\lambda)^2 xy = 36\lambda^2 xy \]
\[ xy( (2-10\lambda)^2 - 36\lambda^2 ) = 0 \]

\[ 4xy (16\lambda^2 - 10\lambda + 1) = 0 \]

Gives \( x = 0 \) or \( y = 0 \) or \( 16\lambda^2 - 10\lambda + 1 = 0 \)

By \( 3 \): \( x = 0 \) or \( y = 0 \).

- \( \lambda = 0 \): \[ y = 0 \]. \((0,0) \) **NOT ON CURVE**
- \( y = 0, \lambda = 0 \) **NOT ON CURVE**
\[ 16x^2 - 10y + 7 = 0 \]

By \( \lambda = 0 \) or \( \lambda = \frac{1}{2} \) by Quadratic Formula:

\[ \lambda = \frac{1}{2} \]

By 3 \(-3x = 3y \Rightarrow y = -x\)

From constraint \( 5x^2 + 5y^2 + 6xy = 1 \)

Get \( 4x^2 = 1 \)

\[ x = \pm \frac{1}{2} \]

Get \( (x, y, \lambda) = (\pm \frac{1}{2}, \pm \frac{1}{2}, \frac{1}{2}) \) \( f = \frac{1}{2} \) (17 marks)

\[ \lambda = \frac{1}{2} \]

By 3 \( \frac{3}{4} x = \frac{3}{4} y \Rightarrow y = x \)

By 5 \( 16x^2 = 1 \)

\[ x = \pm \frac{1}{4} \]

Get \( (x, y, \lambda) = (\pm \frac{1}{4}, \pm \frac{1}{4}, \frac{1}{8}) \) \( f = \frac{1}{8} \) (9 marks)
Find max/min of \( f(x,y) = y e^{-x^2} \)
on ellipse \( 4x^2 + 9y^2 = 1 \)

**Geometric Method**

\[ y e^{-x^2} = c \implies y = c e^{-x^2} \]

**Algebraic Method**
\[ ye^{-x^2} = 8 \lambda x \quad (1) \]
\[ e^{-x^2} = 18 \lambda y \quad (2) \]
\[ 4x^2 + 9y^2 = 1 \quad (3) \]

\( 3 \) \( \implies \) \( \frac{18\lambda y}{-8\lambda x} = \frac{18\lambda y^2}{-8\lambda x} \lambda = 0 \quad (4) \)

\( 4 \) gives \( \lambda = 0 \) or \( 9y^2 = 4x \)
Δ = 0
By (2) get $e^2 = 0$ no solutions

$9y^2 = 4x$

By (3) $4x^2 + 4x = 1$

$x = \frac{-1 \pm \sqrt{2}}{2}$

Since $x = \frac{9}{4}y^2 \geq 0$ only have

$x_* = \frac{-1 + \sqrt{2}}{2}$

By (3) $y_* = \frac{\pm \sqrt{2(2\sqrt{2} - 2)}}{3}$

$f(x_*, y_*) = \frac{\pm \sqrt{2} \sqrt{\sqrt{2} - 1}}{3} \cdot \frac{-1 + \sqrt{2}}{2}$