15.1 DOUBLE INTEGRALS OVER RECTANGLES

VOLUMES OF SOLIDS

Find volume of solid region above rectangle \( R = [a, b] \times [c, d] \) in xy-plane and below graph of a function \( z = f(x, y) \) (where \( f \geq 0 \)).

\[ R = [a, b] \times [c, d] \]

\[ z = f(x, y) \]

\[ \int \int_R f(x, y) \, dA \]

\[ V = \int_a^b \int_c^d f(x, y) \, dy \, dx \]

IDEA: Approximate \( S \) as union of thin boxes with base area \( \Delta A = \Delta x \Delta y \) and height given by value of \( f \) at some point (e.g., top right corner) of each base of each box.

ONE SUCH BOX

\[ f(x_5, y_2) \]

\[ y_0 = c \]

\[ x_0 = a \]

\[ y_1, y_2, \ldots, y_s \]

\[ (x_5, y_2) \]

\[ (x_3, y_2) \]

\[ (x, y) \]
Use \( N_x \) boxes in \( x \)-dim of width \( \Delta x = \frac{b-a}{N_x} \)

\( N_y \) — \( y \)-dim — \( \Delta y = \frac{d-c}{N_y} \)

**Def**: The **double integral** of \( z = f(x,y) \) over rectangle \( R \) is

\[
\begin{align*}
    \int_{R} f(x,y) \, dx \, dy &= \lim_{N_x \to \infty, N_y \to \infty} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} f(x_i, y_j) \Delta A \\
    &= \int_{R} \int_{R} f(x,y) \, dx \, dy
\end{align*}
\]

**Thm**: If \( f \) is CTS then this limit exists.

We can approximate \( \int f \, dA \) using finite values of \( N_x, N_y, R \)

**Ex**: \( R = [0,1] \times [0,1] \)

\[
    z = f(x,y) = 3x^2 + 2y^2
\]

\( N_x = 3, \ N_y = 2 \) — **Top Right Corners**
$$\iint (3x^2 + 4y^2) \, dA = \frac{1}{3} \cdot \frac{1}{2} \left( f \left( \frac{1}{3}, \frac{1}{2} \right) + f \left( \frac{2}{3}, \frac{1}{2} \right) \right)$$
$$+ f \left( \frac{1}{n}, \frac{1}{2} \right) + f \left( \frac{1}{2}, \frac{1}{2} \right) + f \left( \frac{2}{3}, \frac{1}{2} \right)$$
$$+ f \left( \frac{1}{n}, \frac{1}{2} \right)$$
$$= 4.0556.$$
Meanings of \( \delta \int f \, dA \)

1. If \( \rho(x,y) = \text{density of rectangular plate} \ (\text{in} \ \text{kg/m}^2) \), then

\[
R \int_{A} \rho \, dA = \text{Total Mass of Plate}
\]

\[
\text{kg/m}^2 \times m^2 = \text{kg}
\]

2. Average of \( \rho \) = \( \overline{\rho} = \frac{1}{\text{Area}(R)} \int_{A} \rho \, dA \)

If \( \rho \geq 0 \) then

\[
\overline{\rho} \cdot \text{Area}(R) = \int_{A} \rho \, dA
\]

Height \times \text{Area}(R) = \text{Volume of } L

So \( \overline{\rho} \) is height of a box whose volume equals volume of solid \( S \) under \( \rho \)-fog and over \( R \).