MATH 2415 Final Exam, Fall 2014

No books or notes! You may use a scientific calculator provided it does not allow for access to the internet. Show all work and give complete explanations. Don’t spend too much time on any one problem. This 2 hours 45 mins exam is worth 100 points.

(1) [6 pts]
(a) Find the area of the triangle with vertices (1, 2, 3), (2, 0, 3), and (0, 2, 0).

(b) Find a so that the line \( r(t) = (2 - t, 2 + at, 4 + 3t) \) is perpendicular to the plane \( 2x + 3y - 6z = 1 \).
(2) [6 pts] Suppose $z = f(x, y)$ is a function such that $f(1, 2) = 3$, $\frac{\partial f}{\partial x}(1, 2) = 4$ and $\frac{\partial f}{\partial y}(1, 2) = -5$. Use a tangent-plane approximation to the graph of $f$ to estimate the value of $f$ at the point $(0.9, 2.2)$.

(3) [6 pts] Suppose that $(x, y, z) = \mathbf{r}(t)$ is a parametrized curve whose speed is constant. Show that the acceleration vector of the curve is always perpendicular to the velocity vector of the curve.
(4) [12 pts] Find the absolute maximum and absolute minimum of the function \( f(x, y) = x + y - xy \) on the triangle in the \( xy \)-plane with vertices \((0, 0)\), \((4, 0)\), and \((0, 2)\).
(5) [12 pts] The intersection of the surfaces $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$ is a pair of curves.
(a) Find a parametrization for one of these curves.

(b) Show that the curve you found in (a) lies in a plane.

(c) Make a single sketch in three-dimensional space which shows the surface $y^2 + z^2 = 1$, the curve you found in (a), and the plane you found in (b).
(6) [10 pts] Let $E$ be the solid region bounded by the surfaces $x = 0$, $y = 0$, $z = 0$, $y = 1 - x^2$, and $x + y + z = 5$. Find a function $g(x)$ and numbers $a$ and $b$ so that $\iiint_E x \, dV = \int_a^b g(x) \, dx$. 
(7) [10 pts]
(a) Let $C$ be the line segment from $(1, 0, 0)$ to $(4, 1, 2)$ and let $\mathbf{F}(x, y, z) = z^2 \mathbf{i} + y^2 \mathbf{k}$. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

(b) Let $C$ be the circle of radius 3, center $(0, 0)$, oriented clockwise and let $\mathbf{F}(x, y) = y^3 \mathbf{i} - x^3 \mathbf{j}$. Use Green’s Theorem to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$. 
(8) [8 pts] Use the Change of Variables Theorem to find the area of the ellipse \((\frac{x}{a})^2 + \frac{y}{b})^2 = 1\). Hint: Let \((x, y) = (au, bv)\).

(9) [6 pts] Evaluate \(\iiint_E z\, dV\), where \(E\) is the solid hemisphere \(x^2 + y^2 + z^2 \leq 4\), \(z \geq 0\).
Let $S$ be the parametrized surface $x = \frac{1}{2}u \cos v$, $y = u \sin v$, $z = u^2$.

(a) Find a function $f$ so that $S$ is the graph of $z = f(x, y)$.

(b) Sketch the surface $S$ together with the curve on $S$ where $u = 1$ and the curve on $S$ where $v = \pi/4$.

(c) Find a parametrization of the tangent plane to $S$ at the point where $(u, v) = (1, \pi/4)$. 
(11) [7 pts] Let $S$ be the part of the plane $x + y + z = 4$ in the first octant (i.e., where $x > 0$, $y > 0$, and $z > 0$). Calculate $\iint z \, dS$. 
(12) [8 pts] Let \( \mathbf{r} \) be the position vector field, \( \mathbf{r}(x, y, z) = xi + yj + zk \).

(a) Calculate \( \nabla \cdot \mathbf{r} \)

(b) Calculate \( \nabla \times \mathbf{r} \)

(c) Let \( r = |\mathbf{r}| \). Calculate \( \nabla \cdot (\nabla r^2) \).

Pledge: *I have neither given nor received aid on this exam*

Signature: ________________________________