MATH 251 (Fall 2010) Exam II, Oct 21st

No calculators, books or notes! Show all work and give complete explanations. This 65 min exam is worth 50 points.

(1) [6 pts] Suppose that

\[ x = 3u + 2v, \quad y = 4u - 5v \]

and let \( z = f(x, y) \) be a function so that

<table>
<thead>
<tr>
<th>((a, b))</th>
<th>(f(a, b))</th>
<th>(\frac{\partial f}{\partial x}(a, b))</th>
<th>(\frac{\partial f}{\partial y}(a, b))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1, 2))</td>
<td>5</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>((7, -6))</td>
<td>-1</td>
<td>-5</td>
<td>7</td>
</tr>
</tbody>
</table>

Find \( \frac{\partial z}{\partial u} \) and \( \frac{\partial z}{\partial v} \) at \((u, v) = (1, 2)\).

\[ z = f(x(u,v), y(u,v)) \]

\[
\frac{\partial z}{\partial u} (1, 2) = \frac{\partial z}{\partial x} (x(1,2), y(1,2)) \frac{\partial x}{\partial u} (1,2) + \frac{\partial z}{\partial y} (x(1,2), y(1,2)) \frac{\partial y}{\partial u} (1,2)
\]

\[ = \frac{\partial z}{\partial x} (7, -6) \cdot 3 + \frac{\partial z}{\partial y} (7, -6) \cdot 4 \]

\[ = -5 \times 3 + 7 \times 4 = 13 \]

\[
\frac{\partial z}{\partial v} (1, 2) = \frac{\partial z}{\partial x} (7, -6) \frac{\partial x}{\partial v} (1,2) + \frac{\partial z}{\partial y} (7, -6) \frac{\partial y}{\partial v} (1,2)
\]

\[ = 2x(-5) + 7 \times (-5) = -45 \]
(2) [12 pts]
(a) Sketch the parametrized curve \((x, y) = r(t) = (2 \sin t, 3 \cos t)\) for \(0 \leq t \leq \pi\).

\[
x = 2\sin t, \quad y = 3\cos t
\]

So \(\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1\).

Curves is \(\text{part of an ellipse}\)

\[
\begin{align*}
\left(\frac{0}{2}\right) &= (0, 3) \\
\left(\frac{\pi}{2}\right) &= (2, 0) \\
\left(\frac{\pi}{1}\right) &= (0, -3)
\end{align*}
\]

(b) Sketch the level curves of the function \(z = f(x, y) = x - e^y\) at levels \(k = -1, k = 0, \text{and } k = 1\). Also calculate the gradient of \(f\) at the origin, add it to your sketch, and explain how it is related to the level curve that passes through the origin.

\[
z = f(x, y) = x - e^y
\]

\[
z = k \quad x - e^y = k
\]

\[
x = x - k
\]

\[
y = \ln(x - k)
\]

Shift \(y = \ln x\) to right by \(k\).

\[
\nabla f(x, y) = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} = 1 \hat{i} - e^y \hat{j}
\]

\[
\nabla f(0, 0) = 1 \hat{i} - 1 \hat{j} = \hat{i} - \hat{j} = (1, -1)
\]

\(\nabla f(0, 0)\) is perpendicular to the tangent line to the level curve of \(z = f(x, y)\) through \((0, 0)\).
(3) [10 pts] Let \( z = f(x, y) = x^2 + y^3 + 4xy \).

(a) Suppose that the function \( z = f(x, y) \) is temperature at the point \((x, y)\) in the plane. Suppose that a stink bug is walking at constant speed in this plane. In what direction should the stink bug walk from the point \((x, y) = (-1, 2)\) to decrease its temperature the fastest?

\[
\nabla f(x, y) = (2x + 4y, \ 3y^2 + 4x)
\]

\[
\nabla f(-1, 2) = (-2 + 8, 12 - 4) = (6, 8)
\]

Direction \( \mathbf{u} = -\frac{\nabla f(-1, 2)}{||\nabla f(-1, 2)||} = \frac{(-6, -8)}{\sqrt{36 + 64}} = \left(\frac{-3}{5}, \frac{-4}{5}\right)
\]

(b) Find the rate of change of \( f \) at the point \((x, y) = (-1, 2)\) in the direction of the vector \(2\mathbf{i} + 3\mathbf{j}\).

Let \( \mathbf{u} = \frac{2\mathbf{i} + 3\mathbf{j}}{||2\mathbf{i} + 3\mathbf{j}||} = \frac{(2, 3)}{\sqrt{13}} \)

Rate of change of \( f \) in direction \( \mathbf{u} \) at \((-1, 2)\)

\[ = \mathbf{u} \cdot \nabla f((-1, 2)) = \nabla f((-1, 2)) \cdot \mathbf{u} = (6, 8) \cdot \frac{(2, 3)}{\sqrt{13}} = \frac{36}{\sqrt{13}} \]

(c) Find a vector that is tangent to the level curve \( x^2 + y^3 + 4xy = 1 \) at the point \((x, y) = (-1, 2)\).

To find \( \mathbf{v} \) so that \( \mathbf{v} \) is tangent as in picture:

Use fact \( \nabla f(-1, 2) \cdot \mathbf{v} = 1 \) to level curve.

So \( \mathbf{v} \cdot \nabla f(1, 2) = 0 \) must hold.

\[ \nabla f(-1, 2) = (6, 8) \] does the job:

\[ (8, 6) \]

\[ \mathbf{v} = (-8, 6) \] does the job.
(4) [12 pts] Let $S$ be the surface parametrized by

$$\mathbf{r}(u, v) = (1 + \cos u, \sin u, v) \quad 0 \leq u \leq \frac{\pi}{2}, \quad 0 \leq v \leq 4.$$ 

(a) Find a level-set equation of the form $F(x, y, z) = 0$ that is satisfied by all points on $S$.

$$x = 1 + \cos u$$
$$y = \sin u$$
$$z = v$$

$$(x-1)^2 + y^2 = \cos^2 u + \sin^2 u = 1$$

$$F(x, y, z) = (x-1)^2 + y^2 - 1.$$ 

(b) Calculate the tangent vectors to the grid curves $u = \pi/4$ and $v = 2$ at the point $\mathbf{r}(\pi/4, 2)$.

\[
\frac{\partial \mathbf{r}}{\partial u} = (\sin u, \cos u, 0) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \quad \text{at} \quad \pi/4, 2
\]

\[
\frac{\partial \mathbf{r}}{\partial v} = (0, 0, 1)
\]

(c) Sketch $S$ together with the grid curves $u = \pi/4$ and $v = 2$ and their tangent vectors at $\mathbf{r}(\pi/4, 2)$. 
(5) [10 pts] Find all local maxima, local minima, and saddle points of the function \( z = f(x, y) = xy e^y \).

\[
\begin{align*}
z = f(x, y) &= xy e^y \\
\nabla f &= (y e^y, x(e^y + ye^y)) \\
&= (y e^y, x(1+y)e^y) = (0, 0)
\end{align*}
\]

At \( y = 0 \) and \( x = 0 \), \((0, 0)\)

\[
D = \det \begin{bmatrix} 0 & (1+y)e^y \\ (1+y)e^y & x(2+y)e^y \end{bmatrix} = \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1
\]

At \((0, 0)\)

So, Saddle point at \((0, 0)\)

As \( D = -1 < 0 \)

Pledge: I have neither given nor received aid on this exam

Signature: ________________________________