MATH 251 (Fall 2011) Exam III, Nov 22nd

No calculators, books or notes! Show all work and give complete explanations. This 65 min exam is worth 50 points.

(1) [8 pts] Let \( C \) be the straight line segment in the \( xy \)-plane from the point \((1,2)\) to the point \((5,3)\). Let \( \mathbf{F} \) be the vector field in the plane defined by \( \mathbf{F}(x,y) = \frac{1}{2}(xi + yj) \).

(a) Make a sketch showing the vector \( \mathbf{F}(x,y) \) at three points \((x,y)\) on \( C \). Using your sketch, determine whether \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is positive, negative, or zero. Explain!

The angle between \( \mathbf{F} \) and the tangent vector \( \frac{\mathbf{T}}{||\mathbf{T}||} \) to \( C \) is always acute. So

\[
\mathbf{F} \cdot \frac{\mathbf{T}}{||\mathbf{T}||} = ||\mathbf{F}|| ||\mathbf{T}|| \cos \theta > 0
\]

(as \( -\pi/2 < \theta < \pi/2 \))

So \( \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (\mathbf{F} \cdot \mathbf{T}) \, ds > 0 \).

(b) Now calculate \( \int_C \mathbf{F} \cdot d\mathbf{r} \).

\[
\mathbf{T}(t) = (1,2) + t ((5,3) - (1,2)) = (1,2) + t (4,1)
\]

\[
\mathbf{T}(t) = (4t,1) \quad 0 \leq t \leq 1
\]

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2} \int_0^1 (1 + 4t, 2 + t) \cdot (4,1) \, dt
\]

\[
= \frac{1}{2} \int_0^1 6 + 11t \, dt = \frac{29}{4}
\]
(2) [10 pts]
(a) Let $D$ be the half-disc in the $xy$-plane given by $x^2 + y^2 \leq 9$ and $x \geq 0$. Calculate $\iint_D e^{-(x^2+y^2)} \, dA$.

\[
\iint_D e^{-(x^2+y^2)} \, dA \quad \theta = \frac{\pi}{2}, \quad r = 0
\]

\[
\int_{\pi/2}^{3} e^{-r^2} \, r \, dr \, d\theta = \int_{\pi/2}^{\pi} \left( \int_{0}^{3} e^{-r^2} \, r \, du \right)\left( \int_{0}^{\pi/2} \, d\theta \right)
\]

\[
u = -r^2, \quad du = 2r \, dr
\]

\[
= \frac{\pi}{2} \int_{0}^{9} e^{-u} \, du = \frac{\pi}{2} \left[-e^{-u}\right]_{0}^{9} = \frac{\pi}{2} (1-e^{-9})
\]

(b) Let $D$ be the region in the first quadrant (i.e., $x \geq 0$ and $y \geq 0$) of the $xy$-plane that is bounded by the $y$ axis and the curves $y = \sin x$ and $y = \cos x$, and such that $x \leq \pi/4$. Calculate $\iint_D y \, dA$.

\[
\iint_D y \, dA \quad \theta = \frac{\pi}{2}, \quad r = 0
\]

\[
\int_{0}^{\pi/2} \int_{0}^{\cos x} y \, dy \, dx = \int_{0}^{\pi/4} \left[ \frac{1}{2} y^2 \right]_{0}^{\cos x} \, dx
\]

\[
= \frac{1}{2} \int_{0}^{\pi/4} \cos^2 x - \sin^2 x \, dx = \frac{1}{2} \int_{0}^{\pi/4} \cos 2x \, dx
\]

\[
= \frac{1}{2} \left[ \sin 2x \right]_{0}^{\pi/4} = \frac{1}{4}
\]
(3) [10 pts] Let \( r(t) = (2 \cos t, 3 \sin t) \), for \( 0 \leq t \leq 2\pi \), and let \( (u, v) = F(x, y) = (3x + 2y, x^2 + 5y^2) \). The composition \( s(t) = F(r(t)) \) is a curve in the plane. Use the Chain Rule from Multivariable Calculus to answer the following two questions.

(a) At which times, \( t \), is the tangent vector to the curve \((u, v) = s(t)\) vertical?

T.V. is vertical when \( s'(t) = u'(t)i + v'(t)j = \alpha j \) for some scalar \( \alpha \), i.e., when \( u'(t) = 0 \).

Now by Chain Rule

\[
0 = \frac{du}{dt} = \frac{du}{dx} \frac{dx}{dt} + \frac{du}{dy} \frac{dy}{dt}
\]

\[
= 3(-2\sin t) + 2(3 \cos t)
\]

\[
= 3(-2sint) + 2(3 \cos t)
\]

\[
0 = 6(\cos t - \sin t)
\]

So \( \cos t = \sin t \) \( \implies \tan t = 1 \), \( t = \frac{\pi}{4}, \frac{5\pi}{4} \)

(b) For each of the times you found in (a), is the tangent vector pointing in the +j or -j direction?

T.V. in +j direction \( \implies \frac{dv}{dt} > 0 \)

\[
\frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} + \frac{dv}{dy} \frac{dy}{dt}
\]

\[
= 2(2\cos t)(-2\sin t) + 10(3\sin t)3\cos t
\]

\[
= 82 \cos t \sin t
\]

At \( t = \frac{\pi}{4} \), \( \frac{dv}{dt} = 82 \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} > 0 \) \( +j \)

At \( t = \frac{5\pi}{4} \), \( \frac{dv}{dt} = 82 (\frac{-1}{\sqrt{2}})(\frac{-1}{\sqrt{2}}) > 0 \) \( +j \)
(4) [12 pts] Let \( z = f(x, y) = x^3 - 12xy + 8y^3 \).

(a) Find a tangent vector to the level curve \( f(x, y) = 5 \) at the point \((1, -1)\).

A tangent vector \( \vec{v} \) to \( f(x, y) = 5 \) at \((1, -1)\) must be \( 1 + \nabla f (1, -1) \).

Now \( \nabla f = \left( 3x^2 - 12y, -12x + 24y^2 \right) \)

\[ \nabla f (1, -1) = (15, 12) \]

So choose \( \vec{v} = (12, -15) \) for example.

(b) Find all local maxima, local minima, and saddle points of \( f \).

**Critical Points (CPTS)**

\[ 0 = 3x^2 - 12y \]
\[ 0 = -12x + 24y^2 \]

\[ \Rightarrow x^2 = 4y \quad \boxed{1} \]
\[ \Rightarrow 2y = 2y^2 \quad \boxed{2} \]

So by \( \boxed{1} \) and \( \boxed{2} \):

\[ 4y + 4 = x^2 - 4y \]
\[ \Rightarrow y (y^3 - 1) = 0 \]
\[ \Rightarrow y = 0 \text{ or } y^3 = 1 \]
\[ \Rightarrow y = 0 \text{ or } y = 1 \]

Now \( y = 0 \Rightarrow x = 0 \) by \( \boxed{1} \)
\[ y = 1 \Rightarrow x = 2 \] by \( \boxed{2} \)

**CPTS** \((0, 0), (2, 1)\)

\[ D = \begin{vmatrix} 6x & -12 \\ -12 & 4y \end{vmatrix} \]

\[ D = \begin{vmatrix} 0 & -12 \\ -12 & 0 \end{vmatrix} = -144 \text{ < 0} \]

Saddle Point

\[ D = \begin{vmatrix} 12 & -12 \\ -12 & 48 \end{vmatrix} = 12 \times 48 - 12 \times 12 > 0 \]

So \( f(x) = 12 > 0 \) at \( x = 2 \)

Local Min.
Let \( z = f(x, y) \) be a function such that

\[
\begin{array}{c|ccccc}
(x, y) & (2, 1) & (-2, -1) & (0, \sqrt{3}) & (3, 0) \\
\hline
\frac{\partial f}{\partial x} & -10 & 10 & 0 & 4 \\
\frac{\partial f}{\partial y} & -2 & 4 & 0 & -3 \\
\end{array}
\]

Which of the \((x, y)\) values in this table are candidates for the absolute maximum and absolute minimum of \( f \) on the curve \( 2x^2 - 3xy + 4y^2 = 6 \)? Carefully justify your answers!

This is a constrained optimization problem. So candidates are solutions of Lagrange multipliers equations:

\[
\begin{align*}
\nabla f &= \lambda \nabla g \\
g &= 2x^2 - 3xy + 4y^2 = 6
\end{align*}
\]

Now

\[
\frac{\partial f}{\partial x} = 4x - 3y \\
\frac{\partial f}{\partial y} = -3x + 2y
\]

<table>
<thead>
<tr>
<th></th>
<th>(2, 1)</th>
<th>(-2, -1)</th>
<th>(0, \sqrt{3})</th>
<th>(\sqrt{3}, 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial f}{\partial x} )</td>
<td>5</td>
<td>-5</td>
<td>-3\sqrt{3}</td>
<td>4\sqrt{3}</td>
</tr>
<tr>
<td>( \frac{\partial f}{\partial y} )</td>
<td>2</td>
<td>-2</td>
<td>8\sqrt{3}</td>
<td>-3\sqrt{3}</td>
</tr>
<tr>
<td>( g = 2x^2 - 3xy + 4y^2 = 6 )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
</tr>
<tr>
<td>( \nabla f = \lambda \nabla g )</td>
<td>( \times )</td>
<td>( \checkmark )</td>
<td>-</td>
<td>( \checkmark )</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>-2</td>
<td>( \sqrt{3} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Pledge: I have neither given nor received aid on this exam.

Signature: ____________________

So candidates are \((-2, -1)\) and \((\sqrt{3}, 0)\).