Terminology: Let $\mathcal{L}$ be a differential operator, mapping functions to functions. [For example, $\mathcal{L} = \frac{d}{dx}$, in which case $\mathcal{L}(f) = f'$.] We say $\mathcal{L}$ is linear if $\mathcal{L}(u + v) = \mathcal{L}(u) + \mathcal{L}(v)$ and $\mathcal{L}(cu) = c\mathcal{L}(u)$, for all functions $u, v$, and all constants, $c$. An equation of the form $\mathcal{L}(u) = 0$ is called a homogeneous linear equation if $\mathcal{L}$ is a linear operator.¹ An equation of the form $\mathcal{L}(u) = g$, where $g$ is a non-zero function and $\mathcal{L}$ is a linear operator, is called an inhomogeneous linear equation.

(1) [WS] For each of the following equations, state the order, and whether the equation is nonlinear, linear inhomogeneous, or linear homogeneous. (Either prove linearity or provide a simple counterexample showing that the equation is not linear.)

(a) $u_t - u_{xx} + 1 = 0$
(b) $u_t - u_{xx} + xu = 0$
(c) $u_t - u_{xxt} + uu_x = 0$
(d) $u_{tt} - u_{xx} + x^2 = 0$
(e) $u_x + e^yu_y = 0$

(2) [WS] Verify that $u(x, y) = f(x)g(y)$ is a solution of the PDE $uu_{xy} = u_xu_y$ for all pairs of differentiable functions $f$ and $g$ of one variable.

(3) [WS] Verify that $u(x, y) = \sin rx \sinh ry$ is a solution of Laplace’s equation $u_{xx} + u_{yy} = 0$ for every $r \in \mathbb{R}$.

(4) Let $z = f(x, y)$ be differentiable and let $(x_0, y_0)$ be a critical point of $f$. Prove that if $(x_0, y_0)$ is a local maximum of $f$ then $\Delta f(x_0, y_0) \leq 0$. [Hint: Slice the graph of $f$ in the planes $x = x_0$ and $y = y_0$.] Is the converse of this statement true? (Either provide a proof or a counterexample.)

(5) Let $\mathbf{F} = \rho \mathbf{v}$, where $\mathbf{v}$ is the velocity vector field of a fluid flowing in space and $\rho$ is the density of the fluid. Let $S$ be an oriented surface in space with positive unit normal vector field $\nu$. Show that the total mass of fluid crossing $S$ in the positive direction in unit time is given by $\iint_S \mathbf{F} \cdot d\sigma$.

¹So (by definition!), we have that a linear combination (superposition) of solutions of a homogeneous linear equation is another solution of that equation.
(6) [WS] Verify the Divergence Theorem for the vector field \( \mathbf{r}(x, y, z) = xi + yj + zk \) on the ball of radius \( R \) centered at the origin, (i.e., calculate both sides of the equation in the theorem and show they are the same). [A ball is a solid whose boundary surface is a sphere.]

(7) Greens Theorem states that \( \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \oint_{\partial D} P \, dx + Q \, dy \), where \( D \) is a domain in the plane with oriented boundary \( \partial D \). Use Green’s Theorem to prove The Divergence Theorem in \( \mathbb{R}^2 \), which states that for any vector field \( \mathbf{F} \) on \( \mathbb{R}^2 \),

\[
\iiint_D \nabla \cdot \mathbf{F} \, dA = \oint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, ds,
\]
where \( \mathbf{n} \) is the unit outward normal vector field on \( \partial D \).

(8) Give as rigorous a proof as you can of the following **Vanishing Theorem:** Let \( f : [a, b] \to \mathbb{R} \) be a continuous real-valued function on a closed interval. Suppose that \( f \geq 0 \) on \( [a, b] \) and that \( \int f(x) \, dx = 0 \). Then \( f \equiv 0 \) is identically zero.

(9) Give as rigorous a proof as you can of the following **Differentiation Under The Integral Theorem.** Let

\[
I(t) = \int_{a(t)}^{b(t)} f(x, t) \, dx,
\]
where \( f \) and \( \frac{\partial f}{\partial t} \) are continuous and \( a(t) \) and \( b(t) \) are differentiable. Then

\[
\frac{dI}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t} (x, t) \, dx + f(b(t), t)b'(t) - f(a(t), t)a'(t).
\]
[Hint: Express \( I(t) \) as a composition of functions and apply the Chain Rule.]

Additional Questions (Not to be handed in)

(10) Give a careful and complete statement of The Divergence Theorem in \( \mathbb{R}^3 \) from Multivariable Calculus. Include a sketch illustrating the relationships between the various mathematical objects in the theorem. (Feel free to look it up!)

(11) Make sure you understand all details of the proofs of Equations (1.12)-(1.15) of [SS] (page 12).