Additional Problems, Not For Credit

If you want feedback, hand in Problems 1, 2, 3 by Thursday May 2nd in class and I will get them back to you by the following Monday.

(1) [SS, 5.4]

(2) [SS, 5.5]

(3) [SS, 5.17]

(4) [SS, 5.6]

(5) [SS, 5.16]

(6) [MG] Consider the IVP for \( x \in \mathbb{R} \) and \( t > 0 \):

\[
\begin{align*}
    u_{tt} - c^2 u_{xx} &= 0 \\
    u(x, 0) &= g(x) \\
    u_t(x, 0) &= 0
\end{align*}
\]

with \( g(x) = e^{-(x+5)^2} + (-1)^k e^{-(x-5)^2} \) for \( k = 0 \) and \( k = 1 \). The initial displacement consists of two ”blips”, which according to D’Alembert’s formula will each split into a left-moving and right-moving wave. What happens when the right moving wave from the first blip meets the left moving wave from the second blip?

(7) [MG] Consider the IVP for \( x \in \mathbb{R} \) and \( t > 0 \):

\[
\begin{align*}
    u_{tt} - c^2 u_{xx} &= 0 \\
    u(x, 0) &= g(x) \\
    u_t(x, 0) &= h(x)
\end{align*}
\]

Let \([a, b]\) be a closed interval and fix \( x_1 > b \). In each case below find the time intervals during which \( u(x_1, t) = 0 \) holds:

(a) \( g \equiv 0 \) outside \([a, b]\) and \( h \equiv 0 \) everywhere;

(b) \( h \equiv 0 \) outside \([a, b]\) and \( g \equiv 0 \) everywhere.
Find all three-dimensional plane waves, that is, all solutions of the wave equation of the form \( u(x, t) = f(k \cdot x - ct) \), where \( k \) is a fixed vector and \( f : \mathbb{R} \to \mathbb{R} \).

For the three-dimensional wave equation boundary condition \( \frac{\partial u}{\partial \nu} + b \frac{\partial u}{\partial t} = 0 \) on \( \partial \Omega \), with \( b > 0 \) show that the energy

\[
E(t) = \frac{1}{2} \int_{\Omega} u_t^2 + c^2 |\nabla u|^2 \, dx
\]

decreases as time increases.

Why doesn’t the method of spherical means we used to derive Kirchhoff’s Formula work in two-dimensions?

(a) What is the surface area of that portion of the sphere center \( x \) radius \( R \) that lies within the sphere center \( 0 \) radius \( \rho \)? Hint: Divide into cases depending on whether on sphere contains the other or not. Use the law of cosines.

(b) Solve the 3-d wave equation

\[
\begin{align*}
    u_{tt} - c^2 \Delta u &= 0 \\
u(x, 0) &= 0 \\
u_t(x, 0) &= h(x)
\end{align*}
\]

where \( h(x) = A \) for \( |x| < \rho \) and zero elsewhere. Sketch the regions in space-time that illustrate your answer.

(c) Sketch \( u \) versus \( |x| \) for \( t = \frac{1}{2}, t = 1, t = 2 \) assuming \( \rho = A = c = 1 \). (Movie of solution)

(d) Sketch \( u \) versus \( t \) for \( |x| = \frac{1}{2}, t = 2 \) assuming \( \rho = A = c = 1 \). (What a stationary observer sees)

(e) Ride along the wave along a ”light ray” emanating from \((x_0, 0)\). That is plot \( u(x_0 + tv, t) \), where \( |v| = c \). Prove that \( \lim_{t \to \infty} F(t) = tu(x_0 + tv, t) \) exists.