LECTURE 1: INTRODUCTION

MATHEMATICAL MODEL: A set of equations capturing the essential features of a complex natural phenomenon or engineering system.

These equations are

1. Based on
   a. General Laws (of Physics, Chemistry...)
   b. Constitutive Relations
      - Relations between 2 physical quantities that are specific to a particular material
      - E.g., Hooke's Law for Elastic Springs, \( f = -kx \).

2. Typically PDEs or systems of them.

A PDE is an equation involving an unknown function \( u \) of at least 2 variables and its partial derivatives.

E.g., \( u = u(x,y), \quad u_{xx} + u_{yy} = f(x,y) \).
A Pipeline for Computational Science + Engineering Research

1. Develop Mathematical Model
2. Analyze Mathematical Properties of Solutions
   - $J$, $l$, Smoothness
   - Classical Solutions: Formulae for solutions in special cases
3. Numerical Methods for solutions in more general cases
4. Verify that computational code solves equations correctly
5. Validate that equations model phenomenon/system correctly
6. Simulation Studies to
   - Understand + Predict Phenomena
   - Design + Optimize Engineering Systems

Ex Applications
- Weather Forecasting
- Aircraft Design
- Optical Fiber Communications Systems (Internet backbone)
- Heart Dynamics
- Financial Markets
- Galaxy formation + black holes
- Fuel Cells
- Oil exploration + recovery
EXS OF PDES

**Terminology**

The **order** of a PDE is the highest order derivative in the eqn.

A PDE is **linear** if it is a linear equation in $u$ and all its partial derivatives.

**Derivatives**

If $u = u(x,y,z,t)$ then:

- $u_t = \frac{\partial u}{\partial t}$
- $\nabla u = (u_x, u_y, u_z)$ gradient of $u$
- $\Delta u = \nabla \cdot (\nabla u) = \nabla^2 u = u_{xx} + u_{yy} + u_{zz}$ Laplacian of $u$

**Linear PDEs**

1. **Transport Eqn** $u_t + \vec{v} \cdot \nabla u = 0$. (1st order)
   
   - $u$ = Concentration of solid pollutant in channel
   - $\vec{v}$ = fluid velocity

2. **Diffusion (Heat) Eqn** $u_t - D \Delta u = 0$. (2nd order)
   
   - $u$ = Temperature
   - $D$ = Diffusion coefficient of material
3. Wave Eqn \( u_{tt} - c^2 u_{xx} = 0 \)

- \( u \) = Wave amplitude
- \( c \) = Wave speed

Equations:
- Vibration, string, drum membrane, sound, electromagnetic waves.

4. Laplace Eqn \( \Delta u = 0 \)

- Poisson Eqn \( \Delta u = f \)

For time-independent heat diffusion + wave propagation.

5. Helmholtz Eqn \( \Delta u = \lambda u \)

For time-harmonic \( u(x, t) = \psi(x) e^{i\omega t} \)
heat diffusion + wave propagation.

6. Vibrating String Eqn (in 1D) \( u_{tt} + \lambda u_{xx} = 0 \)

Nonlinear PDEs:

7. Burgers' Eqn \( u_t + uu_x = 0 \)

1D Fluid Flow, Traffic Dynamics

8. Eikonal Eqn \( |Du| = c(x, y, z) \)

Level Surface \( u(x, y, z) = t \) is light wavefront at time \( t \).

\[ iut + \Delta u = |u|^2 u \]

For Propagation of light in nonlinear optical material.

10. KdV Eqn \[ u_t + uu_x + u_{xxx} = 0. \]

For Shallow water waves in a channel.

**What does it mean to solve a PDE?**

\[ \text{PDE} \quad \alpha u_x + \beta u_y = 0 \quad \alpha, \beta \in \mathbb{R} \]

\[ u = u(x, y) \]

1. \[ u(x, y) = (bx - ay)^3 \]

Solves PDE since

\[ u_x = 3(bx - ay)^2 b \]
\[ u_y = 3(bx - ay)^2 (-a) \]

So \[ \alpha u_x + \beta u_y = 3(ab - ba)(bx - ay)^2 = 0 \]

2. In fact for any \( f : \mathbb{R} \rightarrow \mathbb{R} \)

\[ u(x, y) = f(bx - ay) \]

Solves PDE.
\[ a u_x + b u_y = a \cdot f' \left( b x - a y \right) \cdot x + b \cdot f' \left( b x - a y \right) \cdot y = \left( a b - b a \right) f' \left( b x - a y \right) = 0. \]

MORAL: Solutions of PDEs generally depend on arbitrary functions.

(Compare with: Solutions of ODEs depend on arbitrary constants.)

So, need initial and/or boundary conditions to obtain a unique soln.

WELL-POSED PROBLEMS

A PDE problem on a domain together with a set of initial and/or boundary conditions is well-posed if

1. There exists at least 1 soln. to problem

2. There is at most 1 soln.

3. Stability: The soln. depends continuously on the data given in the problem. (A small error in data \( \Rightarrow \) small error in soln.)