Lecture 15  

**Burger's Eqn, Shocks, + Traffic Dynamics**

**Inviscid Burger's Eqn**

\[
\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0
\]

with \( q(u) = \frac{u^2}{2} \) gives one of simplest NL Conservative Laws:

\[
\frac{\partial u}{\partial t} + uu_x = 0
\]

\[
\begin{aligned}
\mu(x, 0) &= u_0(x) \\
\end{aligned}
\]

Simple model for some aspects of turbulence, disturbances in 1D fluid flow.

More commonly, used to illustrate shock formation

**Characteristics Eqns**

\[
u = u(x, t)
\]

Find \( x = x(t, s) \)

\[
\dot{x} = y(t, s)
\]

\[
v = v(t, s)
\]
with \[ \begin{align*}
\frac{dx}{dt} &= v \\
\frac{dt}{d\tau} &= 1 \\
\frac{dx}{d\tau} &= 0
\end{align*} \]

and
\[ \begin{align*}
x(s,0) &= s \\
t(s,0) &= 0 \\
v(s,0) &= u_0(s)
\end{align*} \]

\textbf{SOLN}
\[ \begin{align*}
x(s,t) &= s + u_0(s)t \\
t(s,t) &= \tau \\
v(s,t) &= u_0(s)
\end{align*} \]

\[ u_0(s,t) = u_0(s(x,t)) \]

\textbf{Problem}
often can't solve explicitly
\[ x = s + u_0(s)t \quad \text{for} \quad s = s(x,t) \]

\textbf{However}
can be done \textit{in theory} as
\[
J(s, t) = \begin{vmatrix}
\frac{dx}{ds} & \frac{dx}{dt} \\
\frac{dt}{ds} & \frac{dt}{dt}
\end{vmatrix} = \begin{vmatrix}
1 + u_0'(s) & 0 \\
0 & 1
\end{vmatrix}
\]

\[
J(s, t) = 1 + u_0'(s) t
\]

On initial curve \( t = 0 \) in \((s, t)\)-plane we have

\[
J(s, 0) = 1 \neq 0
\]

So \( E \) seen \( u \) for some time interval \([0, T]\)

Chars are curves

\[
(x, t) = (x(s_0, t), t(s, t)) = (s_0 + u_0(s_0) t, t)
\]

\[
x = x_0 + u_0(x_0) t \quad \text{for fixed } x_0
\]

or

\[
t = \frac{x - x_0}{u_0(x_0)}
\]

Line thru \((x_0, 0)\), slope \( m = \frac{1}{u_0(x_0)}\)
Case \( u'_0 > 0 \) for all \( t \geq 0 \)

\[ \text{So } u_0 \uparrow, \ m \downarrow \text{ as } \uparrow \]

\[ J(s, t) > 0 \ \forall t \geq 0 \]

\[ \text{So } u \notin \forall t \geq 0 \]

\[ \mu \text{ is constant along char} \]

Case \( u'_0 \leq 0 \)

\[ u_0 \downarrow, \ m \uparrow \text{ as } \uparrow \]

\[ J(s, t) = 0 \text{ at } t = \frac{-1}{u'_0(s)} > 0 \]

Smallest \( t \) at which \( J(s, t) = 0 \) is

\[ t^* = \min \left( \frac{-1}{u'_0(s)} \right) \text{ s.t. } \exists s \]

\[ t \]

\[ t^* \]

\[ \text{So } u \text{ only } \exists \]

For \( t \in [0, t^*] \)

At \( t = t^* \) \exists 2 \text{ chars that cross}

\( \mu \) becomes multi-valued at \( t = t^* \)
\[ u_0(x) = \frac{1}{1+e^x} \]

For \( x > 0 \), \( u_0(x) \) is decreasing.

The CC meets \( x = 1/\beta \) at the earliest crossing point.

At \( x_0 = 0 \), \( x = \frac{3}{4}t \) with slope 1.

At \( x_0 = \frac{1}{\sqrt{3}} \), \( x = \frac{1}{\sqrt{3}} + \frac{3}{4}t \) with slope \( 4/3 \).

At \( x_0 = -\frac{1}{\sqrt{3}} \), \( x = -\frac{1}{\sqrt{3}} + \frac{3}{4}t \) with slope \( 4/3 \).

The \( x_0 = 0 \) and \( x = 1/\beta \) CCs meet at \( t = \frac{3}{2}t^* \).

Slopes increase as \( x \) decreases, and vice versa.

\[ t^* = \frac{8}{3\sqrt{3}} \approx 1.54 \]
Notice: $u_t + uu_x = 0$

So, like $u_t + cu_x = 0$

So, speed = height

Higher parts of wave travel faster and peak # catches up with right part of wave.

Wave breaks and solution DNE in classical sense.
WHAT IS DOMAIN OF $\mathcal{F}$ OR $\mathcal{U}$? Look at where CC intersects

**IF** $u_0(x_0) > 0$ **CC** **then** $(x_0, 0)$ never intersects another

and **so** $F$ **at least** **that** **CC**

**IF** $u_0(x_0) < 0$ **so** **only** $F$ **on** **CC** **then** $(x_0, 0)$

up to

$$t = \frac{-1}{u_0'(x_0)}$$

Since **CC** $x = x_0 + u_0(x_0) t$

**the** $(x,t)$ **value** **so** **ceases to** $\mathcal{F}$ **is**

$$x = x_0 + \frac{u_0(x_0)}{u_0'(x_0)} t = \frac{-1}{u_0'(x_0)}$$

As $x_0$ (**starting point** of **CC**) **varies** this gives us a **curve** $(x, t) = \mathcal{F}(x_0)$ **in** $(x,t)$ - **space**

**called a CAUSTIC** which is **boundary** of domain of $\mathcal{F}$

**For** **Burger's** **eqn**

$$x = \frac{1 + 3x_0^2}{2x_0} \quad t = \frac{(1 + x_0^2)^2}{2x_0} \quad x_0 > 0$$
In shaded region classical solution DNE, although can regard as a multi-valued function. 

Wave Breaking

At each (x, t) inside the caustic, characteristic curves intersect, so solution is triple valued.
**TRAFFIC FLOW**

Model of congested 1D highway

\[ p = p(x,t) = \text{Traffic Density} \quad \text{# cars/length at } (x,t) \]

\[ q = q(x,t) = \text{Traffic Flow} = \text{# Cars/Time passing } (x,t) \]

\[ u = u(x,t) = \text{Car Velocity at } (x,t) \]

**ASSUMPTION**

\[ u = u(p) \]

with \[ u'(p) \leq 0 \] cars slow as density increases

By units

\[ q = pu \]

**ASSUME FURTHER**

\[ u(p) = u_{\text{max}} \left( 1 - \frac{p}{p_{\text{max}}} \right) \]

Law of Corson of Cars:

\[ pt + qx = 0 \]

Given

\[ \left[ pt + c(p)f_x = 0 \right] \]

\[ pt + u_{\text{max}} \left( 1 - \frac{2f}{p_{\text{max}}} \right) f_x = 0 \]
IC \[ p(x, 0) = f(x). \]

Char Eqns \[ \frac{dp}{dt} = 0 \]
\[ \frac{dx}{dt} = c(p) \quad \text{CHAR VELOCITY} \]
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So, Char are straight lines with slope \( \frac{1}{cp} \) and density is constant along them.

Char thru \((x_0, 0)\) is
\[ x = c(f(x_0))t + x_0. \]

EX RED LIGHT TURNS GREEN

\[ f(x) = p(x, 0) = \begin{cases} p_{\text{max}} & x < 0 \\ 0 & x > 0 \end{cases} \]

CHARS
\[ x = u_{\text{max}} (1 - \frac{2f}{p_{\text{max}}})t + x_0 = \frac{c(p)}{c(p)} \]

\[ c(p) = \begin{cases} u_{\text{max}} & f = 0 \\ -u_{\text{max}} & f = p_{\text{max}} \end{cases} \]
So $p(x,t) = \begin{cases} p_{\text{max}} & x < -u_{\text{max}} \\ 0 & x > u_{\text{max}} \end{cases}$

If your car is at $x^* < 0$, then you only start moving at time $t^* = \frac{-x^*}{u_{\text{max}}}$.

So in format, light turned green propagates backward at speed $+u_{\text{max}}$. (Velocity = $-u_{\text{max}}$)

NOTE: Can't use $\Phi$ if $C$ to find solution in 5 as no chess move into 5.

REASON: $IC$ $f \equiv 0 \not\equiv \Phi$.

REASONABLE APPROACH:

Choose $f_\varepsilon \to f$ as $\varepsilon \to 0$.

Solve $\begin{cases} p_\varepsilon t + c(p_\varepsilon) p_\varepsilon x = 0 \\ p_\varepsilon(x,0) = f_\varepsilon(x) \end{cases}$
Let $p_{E} \rightarrow p \quad \infty \quad z \rightarrow 0$.

$\rho_{E} = \rho_{\text{max}}$

UEV (S.S. P169-170) that $\Delta z$ as

$$p(x,t) = \begin{cases} 
\rho_{\text{max}} x \leq -u_{\text{max}} t \\
\frac{\rho_{\text{max}}}{2} \left(1 - \frac{x}{u_{\text{max}} t}\right) - u_{\text{max}} t < x < 2u_{\text{max}} t \\
0 \quad x \geq u_{\text{max}} t
\end{cases}$$

$p$ is constant on $\text{FAN OF LINES}$

$x = c = vt \quad \text{for} \quad |v| < u_{\text{max}}$

$\text{RAREFACTION WAVE}$
**TRAFFIC JAMS**

\[ f(x) = \begin{cases} \frac{1}{8} f_{\text{max}} & x < 0 \\ f_{\text{max}} & x > 0 \end{cases} \]

Expect congestion to propagate back into original traffic.

**EUQ/CTAPS**

\[ x = \frac{3}{4} u_{\text{max}} t + x_0 \quad x_0 < 0 \]

\[ x = -u_{\text{max}} t + x_0 \quad x_0 > 0 \]

\[ u = q'(p) = \frac{3}{4} u_{\text{max}} \]

\[ p = \frac{f_{\text{max}}}{8} \]

**SHOCK**

Instead of multi-valued solutions (see for water wave)

we expect \( q \) to have a jump discontinuity along a curve \( x = s(t) \) **Shock Curve**

**IDEA** Away from shock curve \( f \) is C^1 to expect J PDE to hold. Along shock, \( f \)

has jump discontinuity.

Use this to derive ODE for \( s \).
Fixed. Suppose \( x_1 < s(t) < x_2 \). Then

\[
\frac{d}{dt} \left( \int_{x_1}^{s(t)} p(x,t) \, dx + \int_{s(t)}^{x_2} p(x,t) \, dx \right) = \text{RHS of } (\ast)
\]

Now

\[
\frac{d}{dt} \int_{x_1}^{s(t)} p(x,t) \, dx = \int_{x_1}^{s(t)} \frac{\partial}{\partial t} p(x,t) \, dx + p(s(t), t) \frac{ds}{dt}
\]

\( p^{-}(s(t), t) = \lim_{y \uparrow s(t)} p(y, t) \)

So get

\[
\int_{x_1}^{x_2} \left[ p^+(s(t), t) - p^-(s(t), t) \right] \, ds(t)
\]

\[
= q \left( \rho(x_1, t) \right) - q \left( \rho(x_2, t) \right)
\]

Letting \( x_1 \uparrow s(t) \), \( x_2 \downarrow s(t) \) gives
RANKINE-HUGONIOT CONDITION

$$\frac{ds}{dt} = \frac{q(p^+, t) - q(p^-, t)}{p^+ - p^-}$$

IF \( s(0) = s_0 \) THEN can solve ODE for SHOCK WAVE \( s = s(t) \).

**Shock Speed**

\[ \text{Flux Jump} \]

\[ \text{Density Jump} \]

EX (CONT'D)

\( p^+ = \rho_{\text{max}} \)
\( p^- = \frac{1}{\rho} \rho_{\text{max}} \)

\( q(p^+) = 0 \)
\( q(p^-) = \frac{7}{64} \rho_{\text{max}} \rho_{\text{max}} \)

(From \( q(p) = \rho_{\text{max}} p (1 - \frac{p}{\rho_{\text{max}}}) \))

So

\[ \frac{ds}{dt} = -\frac{1}{8} \rho_{\text{max}} \]

\( s(0) = 0 \)

\[ s(t) = -\frac{1}{8} \rho_{\text{max}} t \]

\( p(x, t) = \begin{cases} \rho_{\text{max}} & x < -\frac{1}{8} \rho_{\text{max}} t \\ \rho_{\text{max}} & x > -\frac{1}{8} \rho_{\text{max}} t \end{cases} \)