THE 1D WAVE EQUATION

The 1D wave equation for \( u = u(x,t) \), \( x, t \in \mathbb{R} \) is

\[
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f(x,t)
\]

The nD wave equation for \( u = u(x,t) \), \( x \in \mathbb{R}^n \) is

\[
\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = f(x,t)
\]

Here \( c = c(x,t) \) = Wave Speed
\( f = \) Source Term

Types of physics

1 + 2: model may describe wave and vibrational phenomena
- Vibration of violin strings and drum membranes
- Acoustic (sound) waves
- Electromagnetic waves (radio, light)
- Water waves
- Elastic waves in solids
- Quantum mechanical vibrations/wave functions
**Types of Waves**

The following functions all satisfy Eq. (3):

1. **Traveling Waves**
   \[ u(x,t) = g(x-ct) \]
   Profile curve \( u(x,0) = g(x) \) moves in \( +x \) dir. at speed \( |c| \) if \( c \) is \( \pm \) without changing shape.

2. **Harmonic Waves** are Traveling Waves of form
   \[ u(x,t) = A \cos(kx - \omega t) \]
   Physics:
   - \( A > 0 \) is wave amplitude
   - \( k \) is wave number = # Periods in optical interval of length 2\( \pi \)
   - \( \lambda = \frac{2\pi}{k} = \text{Wavelength} = \text{Period} \)
. $\omega$ is angular frequency in [RAD/TIME]

. $f = \frac{\omega}{2\pi}$ is frequency = # Periods in unit of time at fixed $x$

  - UNITS: [TIME$^{-1}$]
  - $s^{-1} = Hz = Hertz$

. Wave Speed $c = \frac{\omega}{k}$ in [LENGTH/TIME]

C. **STANDING WAVES**

$$u(x,t) = A \cos(kx - \omega t)$$

Amplitude of $y = \cos kx$

varies periodically in time:

$$A(t) = B \cos(\omega t)$$

**LOCATIONS OF MAX/MIN DON'T CHANGE WITH TIME.**

D. **PLANE WAVES** in $\mathbb{R}^n$: $u(x,t) = f(\frac{x}{c} - \alpha t)$

for some $f: \mathbb{R} \rightarrow \mathbb{R}$, e.g.

Suppose $f(t)$: $\frac{x}{c} - \omega t = C = \text{constant}$

then

Let $P_C = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} / \frac{x}{c} - \omega t = C\}$, $C \in \mathbb{R}$

Then $u$ is constant, with value $f(C)$, on $P_C$. 
Freeze \( t = t_0 \) and choose \( \xi_0 \in \mathbb{R}^n : \ \xi_0 = u t_0 + c \).

Then \( u \) takes value \( f(c) \) on the hyperplane

\[
\xi_0 \cdot (\xi - \xi_0) = 0
\]

"Wave fronts"

Wave moves with speed \( c = \frac{u}{|k|} \)

\( \xi = \text{const by hyperplane} \)

\( E \) **SPHERICAL (CIRCULAR) WAVES**

Fix \( \xi_0 \in \mathbb{R}^n \)

\[
u(\xi, t) = v(p, t) = \begin{cases} e^{iut} v(p) & \text{STANDING SPHERE} \\ v(p - ct) & \text{TRAVELING SPHERE} \end{cases}
\]

with \( p = |\xi - \xi_0| \)
DERIVATION OF WAVE EQUATION FOR "GUITAR STRING"  

TIGHTLY STRETCHED STRING  

\[ T(x, t) = u(x, t) \]

\[ u = \text{VERTICAL DISPLACEMENT FROM REST POSN.} \]

CONCEPTS:

**A1.** AMPLITUDE OF VIBRATION IS SMALL (relative to length of string)

**A2.** Each pt of string undergoes vertical, but not horizontal displacements [So curve given by string is a GRAPH over x axis]

**A3.** String is perfectly flexible - does not resist bending

So FORCE exerted on string is **TENSION** to string.
\[ p = \rho(x, t) = \text{linear density} \]
= mass per unit length as one string
for the point on string with
horizontal position \( x \) at time \( t \)

\( \rho_0(x) \) = density of string at rest

Note

By (4.2) mass of 1 = mass of 2
But length 2 > length 1
So density of 2 < density of 1

String is stretched when
not at rest!!

In fact

**Conservation of mass** with (4.2) says

\[ \rho_0(x) \, dx = \rho(x, t) \, ds = \text{mass over } [x, x + dx] \]

Where  \( ds = \text{length of string over } [x, x + dx] \).

**Newton's Law** Since no horizontal motion of point on string,
the horizontal forces on segment of string from
\( x \) to \( x + dx \) must balance. These tension
(stretching) forces act on either end of segment

\[ |\tau|(x, t) \cos(\theta(x, t)) = |\tau|(x + dx, t) \cos(\theta(x + dx, t)) \]
So \[ \frac{d}{dx} \left[ \left( \frac{1}{T} \right) \cos \theta \right] f(x, t) = 0 \]

\[ \frac{1}{T} \cos \theta = T_0(t) > 0 \]

so constant in \( x \).

**Vertical Motion of String**

Next apply Newton's 2nd Law to segment of string over \([x, x+dx]\):

![Diagram of vertical tension forces]

**Notice**

1. \( T_{\text{vert}}(x, t) = \frac{1}{T} \) \( \sec \theta \)
   \[ = \frac{T_0(t) \tan \theta}{T_0(t) \sec \theta} \]
   \[ = \frac{T_0(t)}{T_0(t)} \frac{du}{dx} \text{ (both are slope!)} \]

2. \( F_{\text{body}} = \int f(y, t) f(y, t) \, ds \)
   \[ = \int_{x+dx}^{x} f(y, t) \, dy \]
\[ \int_{x}^{x+\Delta x} f(y,t) u_{tt}(y) \, ds = \int_{y}^{y+\Delta y} f(y) u_{tt} \, dy \]

"L H S = m a \ \text{in} \ F = m a"

So
\[ \int_{x}^{x+\Delta x} \left[ f(y) u_{tt}(y,t) - \frac{T_0}{V} u_{xx}(x,t) \right] \, dy = 0 \]

Hence
\[ u_{tt} - c^2 u_{xx} = f \]

where
\[ c^2 = \frac{T_0}{f_0(y)} \]

Units of \( c \) = \( \frac{N}{(kgm)} = \sqrt{\frac{N}{(kgm)}} \)

So \( c \) is a speed.

**Note**

1. String Homogeneous \( f_0 = \text{const} \)
2. String Perfectly Elastic \( T_0 = \text{const} \)

(Horiz Tension is same as \( f_0 \) at rest)
So for homogeneous, perfectly flexible, perfectly elastic string, \( c = \text{const} \).

WELL-POSEDNESS

1. A bounded smooth domain in \( \mathbb{R}^n \).
2. PDE \( u_t - c^2 u_{xx} = f \); \( u(x,0) = g(x), \quad u_t(x,0) = h(x) \) on \( \mathbb{R} \).
3. BC
   - Dirichlet: \( u = 0 \) on \( \partial \Omega \).
   - Neumann: \( \frac{\partial u}{\partial n} = 0 \) on \( \partial \Omega \).

CONSERVATION OF ENERGY

\[
E(t) = \frac{1}{2} \int \left( u_t^2 + c^2 |\nabla u|^2 \right) \, dx
\]

\[
\frac{dE}{dt} = KE + \text{Elastic Pot Energy} \quad \text{(up to const)}
\]

IF \( c = \text{constant} \)

\[
\dot{E}(t) = \frac{1}{2} \int \left( u_t u_{tt} + c^2 \nabla u \cdot \nabla u_t \right) \, dx
\]
By Green's Identity
\[ \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \]

So by wave eqn get
\[ E(t) = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{\partial u}{\partial x} \right)^2 + c^2 \left( \frac{\partial u}{\partial t} \right)^2 \, dx \, dt \]

fech. k = f \times ant. sign \Rightarrow k \neq t

The wave eqn with IC and BC as above has at most 1 soln in \( C^{2+2}(\Omega_T) \cap C^{1+1}(\partial_T) \)

PF let \( u_1, u_2 \) be 2 solns, \( u = u_1 - u_2 \).

Then
1. \( f = 0 \) in \( \Omega \)
2. Either \( \frac{\partial u}{\partial \nu} = 0 \) on \( \partial \Omega \)

\[ u = 0 \Rightarrow u_t = 0 \text{ on } \partial \Omega \]

So
\[ E(t) = E(0) = \text{const} \]

Hence
\[ u_t = 0, \quad u = 0 \text{ by BC} \]

So
\[ u = \text{const} = 0 \text{ by BC} \]
SEPARATION OF VARIABLES for "GUITAR STRING"

PDE \[ uu_{tt} - c^2 uu_{xx} = 0 \quad 0 < x < L, \quad t > 0 \]

BC \[ u(0,t) = u(L,t) = 0. \]

IC \[ u(x,0) = g(x), \quad u_t(x,0) = h(x) \]

Can use Separation of Variables to find solution

\[ u(x,t) = \sum_{m=1}^{\infty} \left[ g_m \cos\left(\mu_m x\right) + \frac{h_m}{\mu_m c} \sin\left(\mu_m x\right) \right] \sin(\mu_m t) \]

with \[ \mu_m = \frac{m \pi}{L} \]

\[ g(x) = \sum g_m \sin(\mu_m x) \]

\[ h(x) = \sum h_m \sin(\mu_m x) \]

\[ \sum \] is superposition of simple standing waves.