LECTURE 18  CAUCHY PROBLEM + KIRCHHOFF'S FORMULA IN $\mathbb{R}^3$

SPHERICAL WAVES

Look for spherically symmetric solutions of

$$u_{tt} - c^2 \Delta u = 0 \quad \text{in } \mathbb{R}^3$$  \(1\)

So

$$u(x, t) = w(r, t), \quad r = |x|$$

In spherical coords ($r, \theta, \phi$)

$$D = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left( \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial r^2} \right)$$

So \(1\) becomes

$$w_{tt} - c^2 \left[ w_{rr} + \frac{2}{r} w_r \right] = 0$$  \(2\)

$$w = \frac{(rW)_{tt} - c^2 (rW)_{rr}}{r} = 0$$

So by 1st Member:

$$w(r, t) = \frac{F(r + ct)}{r} + \frac{G(r - ct)}{r}$$  \(3\)

**Superposition of attenuated travelling waves**
THE FUNDAMENTAL SOLN \( (\mathbb{R}^3) \)

\[
\begin{cases}
\partial_t^2 u - c^2 \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\
u(x,0) = 0 \\
\partial_t u(x,0) = S_y(x) = \hat{S}(x-y) \quad \text{INITIAL DISTURBANCE}
\end{cases}
\]

Approx \( S_y \) by F.S. of Heat eqn (with \( t = \varepsilon \))

\[
S_y(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^{\varepsilon} \frac{1}{(4\pi \tau)^{3/2}} e^{-|x-y|^2/4\tau} d\tau
\]

and \( \hat{S} \) by \( \hat{S}(e) \) with \( S_\varepsilon(x,t) = \hat{S}(e) \delta(t) \)

that is spherically symmetric about \( x \):

From above,

\[
\hat{S}(e) \delta(t) = \frac{F(e+c+1)}{r} + \frac{G(e-c)}{r}
\]

II \( \circ \) \( F(e) + G(e) = 0 \implies F = -G \)

\[
\frac{c}{r} \left[ F'(e) - F(e) \right] = \frac{1}{(4\pi \varepsilon)^{3/2}} e^{-r^2/4\varepsilon}
\]

Soln \( \frac{1}{4\pi c \sqrt{4\pi \varepsilon}} \left[ e^{-r^2/4\varepsilon} - 1 \right] \)
So
\[ w_3(x,t) = \frac{1}{4\pi c^2} \int \frac{1}{\sqrt{4\pi \varepsilon t}} e^{-(r - ct)^2/4\varepsilon} e^{-(r + ct)^2} dr \]

Now
\[ \hat{w}_3(r) = \frac{1}{\sqrt{4\pi \varepsilon t}} e^{-r^2/4\varepsilon} \Rightarrow \text{as } t \to \infty \text{ to Heat Eqn when } n = 1 \]

So
\[ w_3(x,t) \to \frac{1}{4\pi c^2} \left[ \delta(r - ct) - \delta(r + ct) \right] \]

So 2nd of 4) is FS of Wave Eqn

\[ k(x,t; \xi) = \frac{\delta(r - ct)}{4\pi c^2} \quad r = |x - \xi| \]

OUTCOME TRAVELLING WAVE INITIALLY CONCENTRATED AT \( y \)

and at that supported on SPIRES

\[ \mathcal{S}_c (\xi) = \delta \delta : / |x - \xi| = c + \delta \]
**Strong Huygen's Principle**

Point source at $y$ at time $0$

Only felt at $x_0$ at time $t = \frac{x_0 - y}{c}$

This is why we can hear what people say.

(This is very different for what happens in $\mathbb{R}^2$)

**Homogeneous Cauchy Problem in $\mathbb{R}^3$**

\[
\begin{cases}
  u_{tt} - c^2 \Delta u = 0 \\
  u(\vec{x},0) = g(\vec{x}) \\
  u_t(\vec{x},0) = h(\vec{x})
\end{cases}
\]

\[
\begin{cases}
  u_{tt} - c^2 \Delta u = 0 \\
  u(\vec{x},0) = 0 \\
  u_t(\vec{x},0) = h(\vec{x})
\end{cases}
\]

Solution $u(\vec{x})$

Non soln of $I \Rightarrow$ sum of solns of II, III
Lemma: If $u^0$ solves (I), then $\mu^0(x_0) = g(x_0)$.

Then $V = (u^0_t)$ solves (III).

PDE:

$V(x_0) = u^0_t(x_0) = g(x_0)$ by (III)

$V_t(x_0) = u^0_{tt}(x_0) = c^2 \Delta u^0_t(x_0) = 0$

as $u^0_t(x_0) = 0$.

Cor: Soln of (I) is of form

$u = \phi_t u^0 + \psi$ (true for $R^3$).

Thm: Kirchhoff's Formula ($R^3$)

Soln of (II) is

$$u(x,t) = ct \int_{S_t(0)} f ds + \frac{1}{ct} \left( ct \int_{S_t(0)} g ds \right)$$
Huygens' Principle

1. Value of $u \text{ at } (x, t)$ only depends on values of $g_1$, $h$ on $S_{ct}(x)$

**Intuition**

- For $ct$ factor
- Height
- Area $= 2\pi ct$, Height

Wavefronts passing thru here at time $0$ reach $x$ at time $t$
Can show $\mathbb{SS}$ that in $\mathbb{R}^2$

$$u(x,t) = \frac{1}{2\pi c} \int \frac{g(x-y) \, dy}{\sqrt{c^2 t^2 - |x-y|^2}}$$

$$+ \int \frac{h(y) \, dy}{B_c(t) \sqrt{c^2 t^2 - |x-y|^2}}$$

where

$$B_c(t) = \text{DISC} = \left\{ y \mid \frac{x-y}{c t} \leq 1 \right\}$$

**Uphost**

**Disturbance** at $y$ at time $0$

reaches $x$ at $t_{\text{min}} = \frac{|x-y|}{c}$

**But effective** persists at $x$ after $t_{\text{min}}$

*Ex drop stone in water*

Cork bobs up & down for a while, not just once
PROOF OF KIRCHHOFF (123) [LE, 2.4]

\[ u_{tt} - c^2 u_{xx} = 0 \]
\[ u(x, 0) = 0 \]
\[ u_t(x, 0) = h(x) \]

Spherical mean of \( u \) over sphere \( S_r(x) \) is

\[
U(x, t; x) = \frac{1}{S_r(x)} \int_{S_r(x)} u(\xi, t) \, d\xi
\]

- \( U(x, t; x) \) describes portion of wave at time \( t \) that propagates to \( x \) at time \( t \).
- Familiar from MVP for harmonic functions.

Also

\[
H(x; t; x) = \frac{1}{S_r(x)} \int_{S_r(x)} h(\xi) \, d\xi
\]

**Lemma**

1. \( \frac{1}{r^2} U_{tt} - U_{rr} - \frac{2}{r} U_r = 0 \) in \( \mathbb{R}^+ \times (0, \infty) \)
2. \( U = 0 \) at \( t = 0 \)
3. \( U_t = H \) at \( t = 0 \)
PF. 1, 2 are immediate from def

2. A m pf of MVP for harmonic f

\[ u_r^c (t; x) = \frac{r}{3} \int_{B_r(x)} p(u) \, dy \]

So

\[ u_r = \frac{r}{3} \int_{B_r(x)} \frac{1}{2} u_{tt} \, dy \]

\[ \frac{r^2 u_r}{u_r} = \frac{1}{4\pi c^2} \int_{B_r(x)} u_{tt} \, dy \]

\[ \frac{\partial}{\partial r} \left( \frac{r^2 u_r}{u_r} \right) = \frac{1}{4\pi c^2} \frac{\partial}{\partial r} \left( \int_{B_r(x)} u_{tt} \, dy \right) \]

\[ = \frac{1}{4\pi c^2} \int_{S_r(x)} u_{tt} \, ds \]

\[ = \frac{1}{c^2 r^2} \frac{1}{2} \left( \frac{r^2 u}{u} \right)_{tt} \]

Let

\[ \frac{1}{r^2} \left[ 2u_r + \frac{r^2 u_{rr}}{u} \right] = \frac{1}{c^2} u_{tt} \]
\[ \mathcal{U} = \mathcal{V} = -\mathcal{H} \]

From Lemma, \( \mathcal{V} \) is:
\[ \begin{align*}
\mathcal{V}_{tt} + c^2 \mathcal{V}_{xx} &= 0 \quad \text{in} \quad \mathbb{R}^+ \times (0, \infty) \\
\mathcal{V} &= 0 \quad \text{on} \quad \mathbb{R} \times [0, \infty) \\
\mathcal{V} &= 0, \quad \mathcal{V}_t = \mathcal{H} \quad \text{on} \quad \mathbb{R}^+ \times (t = 0)
\end{align*} \]

Apply D'Alambert's to get:
\[ \mathcal{V}(r, t; x) = \frac{1}{2\sqrt{c}} \int_{r-c}^{r+c} \mathcal{H}(y) \, dy \]

\[ \mathcal{V}(r, t; x) = \frac{1}{2\sqrt{c}} \int_{ct-r}^{ct+r} \mathcal{H}(y) \, dy \]

Take \( \lim_{r \to \infty} \) to get:
\[ \mathcal{H}(ct) = \int_{ct-r}^{ct+r} \mathcal{H}(y) \, dy \]

\[ = ct \int_{S_{r+c}(c)} \mathcal{H} \, dx \]