LEcTure 4  UUINIQUENESS OF SOlUTIONS OF DIFFUSION EQUATION

Back to Diffusion Eqn in nD

Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain (open, connected set) that models a heat conducting body. Suppose \( \Omega \) is sufficiently smooth.

Let \( \partial \Omega \) be the boundary surface of \( \Omega \) and \( \overline{\Omega} = \Omega \cup \partial \Omega \) the closure of \( \Omega \).

\[
\begin{align*}
\Omega & = \text{Ball} = \{ |x| < 1 \} \\
\partial \Omega & = \text{Sphere} = \{ |x| = 1 \} \\
\overline{\Omega} & = \{ |x| \leq 1 \}
\end{align*}
\]

Study the evolution of temperature in \( \Omega \) over \([0,T]\).

\[
\begin{align*}
PDE & \quad \frac{\partial u}{\partial t} - D \frac{\partial^2 u}{\partial x^2} = f \quad \text{in } \Omega \\
\text{IC} & \quad u(x,0) = g(x), \quad x \in \overline{\Omega}
\end{align*}
\]

\[
\begin{align*}
\text{BC6} \quad \text{DIRICHLET} & \quad u(x,t) = h(x,t) \quad x \in \partial \Omega, \quad 0 < t < T
\end{align*}
\]

\[
\begin{align*}
\text{NEUMANN} \quad \text{let } \mathbf{\nu} \text{ be outward unit normal V.F. on } \partial \Omega.
\end{align*}
\]

\[
\begin{align*}
\text{Heat Flux is } \quad \frac{\partial u}{\partial n} = -K \frac{\partial u}{\partial n} \\
\text{So inward Heat Flux across } \partial \Omega \text{ is } -\frac{\partial u}{\partial n} \cdot \mathbf{\nu} = K \frac{\partial u}{\partial n} \cdot \mathbf{\nu}
\end{align*}
\]
DEF Normal derivative of $u$ on $\partial \Omega$ is

$$\frac{du}{d\nu} = \frac{d}{dt} u = \nabla u \cdot \nu.$$ 

= Directional Derivative of $u$ in the direction $\nu$.

So Neumann conditions are

$$d_u u(x, t) = h(x, t) \quad x \in \partial \Omega, \quad t > 0.$$ 

For Robin and Mixed BCs see [SSS, p28].

**Integral Method for Uniqueness of Solns**

DEF Let $C^{2,1}((\Omega_T)) = \{u: \Omega_T \to \mathbb{R} \text{ with} \}

$$u, \, u_t, \, u_{x_i}, \, u_{x_i x_j} \text{ all cts }, \quad x_i, x_j = 1, 2, 3.$$ 

THM The problem 1, 2 with BCs either

3 or 4 has at most one solution

in $C^{2,1}((\Omega_T))$. 
Let \( u, v \) be 1 such \( \partial u \) and \( w = u - v \).

Then

\[ \text{PDE} \quad w_t - D \Delta w = 0 \quad \text{5} \]

\[ \text{IC} \quad w(x,0) = g(x), \quad x \in \Omega \quad \text{6} \]

and \( \text{DC} \quad w = 0 \quad \text{on} \; \partial \Omega \quad \text{7} \)

or \( \text{NC} \quad \partial_n w = 0 \quad \text{on} \; \partial \Omega \quad \text{8} \)

\[ 5 \times \int \omega \quad \text{and integrate over } \Omega: \]

\[ \int \omega w w_t \, dV = \int \omega w \Delta w \, dV \quad \text{9} \]

Now

\[ \frac{1}{2} \frac{d}{dt} \int \omega w^2 \, dV = \frac{1}{2} \int \frac{\partial}{\partial t} (w^2) \, dV \]

\[ = \int \omega w w_t \, dV \quad \text{9} \]

And

\[ E(t) = \int \omega w^2 \, dV \quad \text{energy of } w, \quad \text{10} \]

Show \( E = 0 \quad \forall t \)

Hence \( w = 0 \quad \forall (x,t) \in \overline{\Omega}_t \).
Let's use integration by parts on \( \int_a^b w \cdot w \, dV \).

In 1D, this is easy:
\[
\int_a^b w \cdot w \, dx = \left[ u w x \right]_a^b - \int_a^b (w x)^2 \, dx,
\]
by BC or NC.

In nD, we need Green's identity:
\[
\int_a^b w \cdot \nabla u \, dV = \int_a^b w \cdot \nabla v \, dV - \int_a^b \nabla w \cdot \nabla v \, dV.
\]

Choose \( u = v = w \) in (11) to get
\[
\int_a^b w \cdot \nabla w \, dV = \int_a^b w \cdot \nabla w \, dV - \int_a^b |\nabla w|^2 \, dV.
\]
by BC or NC.

**Upshot**

By (9) - (12)
\[
\frac{dE}{dt} = -2D \int_a^b |\nabla w|^2 \, dV \leq 0 \text{ for } t > 0.
\]
Also by IC \( E(0) = 0 \).
So $E(t) = 0 \forall t > 0$

So $w(x,t) = 0 \forall x \in \mathbb{R}, \forall t > 0$

So $u = v$.

THE MAXIMUM PRINCIPLE

SPACE-TIME CYLINDER

$Q_T = \Omega \times (0,T)$

PARABOLIC BOUNDARY

$\Gamma_T = \partial \Omega \times (0,T) \cup \partial \mathbb{R} \times (0,T]$  

$\Gamma_T =$ Bottom & Sides.

IN HOMOGENEOUS HEAT BOWL $u_{tt} - Du_{xx} = f(x,t)$

where $f \leq 0$ models a HEAT SINK.

(0 is like a refrigerator!)

The MAX PRINCIPLE says $u$ is largest on $\partial \Omega$ or at $t = 0$, i.e. on $\Gamma_T$. 

\[ \text{\scriptsize \#} \]
Physically, since $f \leq 0$ in $\Omega_T$, heat is being lost there. So, heat flows from $\Omega_T$.

**Thm (Weak Max Principle)**

Let $u \in C^2(\Omega_T) \cap C(\overline{\Omega_T})$ satisfy

$$u_t - \Delta u = f \quad \text{with} \quad f \leq 0 \quad \text{in} \quad \Omega_T$$

Then $u$ attains its $\max$ on $\Gamma_T$:

$$\max_{\overline{\Omega_T}} u = \max_{\Gamma_T} u$$

In particular, if $u < 0$ on $\Gamma_T$, then $u < 0$ on $\overline{\Omega_T}$.

**For**

1. If $u_t - \Delta u = 0$, then $u$ attains its $\max + \min$ at $\Gamma_T$. So

$$\min_{\Gamma_T} u \leq u(x,t) \leq \max_{\Gamma_T} u$$

for all $(x,t) \in \Omega_T$. 
2. Suppose \( v_t - D v = f_1 \), \( w_t - D w = f_2 \)

Then

1. If \( v = w \) on \( \Gamma_T \) and \( f_1 = f_2 \) in \( \Omega_T \)

Then \( v = w \) on all of \( \Omega_T \)

b. **Stability Estimate**

\[
\max_{\Omega_T} |v - w| \leq \max_{\Gamma_T} (v - w) + T \max_{\Omega_T} |f_1 - f_2|
\]

In particular, the initial-Dirichlet problem for heat eqn. has at most 1 soln. This soln. depends continuously on data:

If \( v = g_1 \), \( w = g_2 \) on \( \Gamma_T \) and

\[
\max_{\Gamma_T} |g_1 - g_2| \leq \varepsilon, \quad \max_{\Omega_T} |f_1 - f_2| \leq \varepsilon
\]

Then

\[
\max_{\Omega_T} |v - w| \leq \varepsilon (1 + T)
\]

**Proof of soln:** Huygens
PF of THM

CASE \( f < 0 \)

Since \( \Omega_T \) is closed and bounded and \( u = 0 \) on \( \partial \Omega_T \),
\( u \) attains a max at some \( (\bar{x}, \bar{t}) \in \Omega_T \) in the interior of the space-time cylinder.

(a) Suppose \((\bar{x}, \bar{t})\) is an interior max of \( u = u(x, t) \),
then \( \frac{du}{dt} = 0 \) and \( \nabla u = 0 \) at \((\bar{x}, \bar{t})\).

[CRITERIA PT] and \( \Delta u \leq 0 \) at \((\bar{x}, \bar{t})\).

Review for (13)

Set \( X(x) = u(x, y_0, z_0, t_0) \) slice of \( u \) in \( y = y_0, z = z_0, t = t_0 \)
then \( X \) has max at \( \bar{x}_0 \)

so \( X''(\bar{x}_0) \leq 0 \).

so \( \Delta u(x) = X''(\bar{x}_0) + y''(y_0) + z''(z_0) + T''(t_0) \leq 0 \)

so at \((\bar{x}, t)\)

\( f(\bar{x}, t) = ut - D_t u = 0 - D_t u \geq 0 \).

(b) Suppose \((\bar{x}_0, T)\) is a max on top of space-time cylinder.
Then at \((\bar{x}_0, T)\):
\( \frac{du}{dt} \geq 0 \) by

\[ T \rightarrow \]
and \( \nabla u = 0 \) c. pt \( \in \Omega \times \{T\} \)
and \( \Delta u = 0 \) as \( \max u \in \Omega \times \{T\} \).

So
\[
f(x_0, T) = u_T - D_\Omega u \geq 0.
\]

So \( (x_0, t_0) \in \Gamma^T \) must hold.

\[\text{CASE } f = 0 \]

Let \( \varepsilon > 0 \)

**TRACK** Define \( V = u + \frac{\varepsilon}{2nD} |x|^2 \) for \( x \in \Omega \times [0, T] \)

\[
V(x, t) = \sum_{j=1}^{n} \frac{\partial^2 V}{\partial x_j^2} (x_j^2) = 2n.
\]

So
\[
V_T - D_\Omega V = u_T - D_\Omega u - \varepsilon = f - \varepsilon < 0.
\]

Now apply previous case to conclude
\[
u \leq v \leq \max_{\Omega \times \{T\}} v = \max_{\Gamma^T} \frac{\partial v}{\partial \nu} \leq \max u + \frac{\varepsilon}{2nD} \max_{\Gamma^T} |x|^2
\]
\[
= \max u + \varepsilon M \quad \text{since } \Gamma^T \text{ bounded.}
\]
Taking \( l \rightarrow \infty \) gives

\[ u \leq \max_u \mu \]