Let $x_i; i = 1, ..., N$, be $N$ independent, identically distributed random variables. Assume $E\{x_i\} = \mu$ and $Var\{x_i\} = \sigma^2$ for all $i$. We wish to estimate $\mu$ using the following:

$$\hat{\mu} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

1. Find $E\{\hat{\mu}\}$. Show all the work.

2. Find $E\{\hat{\mu}^2\}$. Show all the work.

3. Find $Var\{\hat{\mu}\}$ using (1) and (2) and comment on $Var\{\hat{\mu}\}$ as $N \to \infty$.

1) $E(\hat{\mu}) = \frac{1}{N} \sum_{i=1}^{N} E(x_i) = \frac{N\mu}{N} = \mu$

2) $E(\hat{\mu}^2) = \frac{1}{N^2} \sum_{i=1}^{N} E(x_i x_j)$

$$= \frac{1}{N^2} \sum_{i=1}^{N} E(x_i) + \frac{1}{N^2} \sum_{i, j} E(x_i x_j)$$

$$= \frac{1}{N^2} \sum_{i=1}^{N} \mu^2 + \frac{1}{N^2} \sum_{i \neq j} \sigma^2$$

$$E(\hat{\mu}^2) = \frac{\sigma^2}{N} + \mu^2$$

3) $Var(\hat{\mu}) = E(\hat{\mu}^2) - [E(\hat{\mu})]^2 = \frac{\sigma^2}{N} + \mu^2 - \mu^2$

$$= \frac{\sigma^2}{N}$$

$N \to \infty \quad Var(\hat{\mu}) \to 0$