Anti-Final

1. A popular web page gets 6 hits per minute, on the average. The hits are modeled by a Binomial counting process with 1-second frames. Calculate:

   (a) The probability of a new hit during any one frame
   (b) The expected time between consecutive hits
   (c) The probability of more than 3 hits during a 20-second span.

Solution.

(a) We have $\lambda = 6 \text{ min}^{-1} = 0.1 \text{ sec}^{-1}$ and $\Delta = 1 \text{ sec}$. Then

$$p = \lambda \Delta = 0.1$$

(b) $E(T) = 1/\lambda = 10 \text{ sec}$

(c) We need $P(X > 3)$, where $X$ is the number of hits in 20 sec. This $X$ has Binomial distribution with parameters

$$n = t/\Delta = 20/1 = 20, \quad p = 0.1.$$ 

From the table of Binomial distribution,

$$P(X > 3) = 1 - F(3) = 1 - 0.8670 = 0.1330$$
2. Cash register in a dairy store represents an M/M/1 queuing system. On the average, 20 customers visit the store every hour, and it takes an average time of 2 minutes for a cashier to serve each of them.

(a) What is the probability that a customer has to wait for the cashier?
(b) What is the expected waiting time before the service begins?
(c) What proportion of customers have at least two customers in front of them, including the customer receiving service?

\[ \text{Solution. } \text{This M/M/1 system has } \lambda_A = 20 \text{ hrs}^{-1} = (1/3) \text{ min}^{-1}, \mu_S = 2 \text{ min}, \text{ and therefore, } \lambda_S = 0.5 \text{ min}^{-1}. \text{ For this system, the arrival-to-service ratio is } r = \frac{\lambda_A}{\lambda_S} = \frac{1/3}{1/2} = \frac{2}{3}. \]

(a) \[ P(W > 0) = P(\text{server is busy}) = r = \frac{2}{3} \text{ or } 0.667 \]

(b) \[ E(W) = \frac{r \mu_S}{1-r} = \frac{(2 \text{ min})(2/3)}{1/3} = 4 \text{ min} \]

(c) \[ P(X \geq 3) = \sum_{x=3}^{\infty} \pi_x = \sum_{x=3}^{\infty} r^x (1-r) = r^3 = \frac{8}{27} \text{ or } 0.296 \]
3. Screening of a computer system by a new antivirus software takes a random time with a standard deviation \( \sigma = 12 \) min. When this software was tried on 40 randomly selected computers, the average screening time was 27 min.

(a) Compute a 95% confidence interval for the expected time it takes to screen a computer for viruses.

(b) At the 5% level of significance, is there a sufficient evidence that the expected screening time is less than 30 min?

**Solution.**

(a) 
\[
\bar{X} \pm z_{0.025} \frac{\sigma}{\sqrt{n}} = 27 \pm \frac{(1.96)(12)}{\sqrt{40}} = 27 \pm 3.72 = [23.28 \text{ min}, 30.72 \text{ min}]
\]

(b) Test \( H_0 : \mu = 30 \) vs \( H_A : \mu < 30 \).

Reject \( H_0 \) if \( Z < -z_{0.05} = -1.645 \).

\[
Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{27 - 30}{12/\sqrt{40}} = -1.5811,
\]

belongs to the acceptance region, so we accept \( H_0 \). No, the data do not provide a sufficient evidence that the expected screening time is less than 30 min.
4. A telephone with only 1 line represents a Bernoulli single-server with capacity \( C = 1 \). On the average, the incoming calls occur every 10 minutes. The average conversation lasts for 5 minutes. Let \( X(t) \) be the number of jobs in the system after \( t \) frames. Assuming 1-minute frames,

(a) Compute the transition probability matrix of \( X \).
(b) Compute the steady state distribution of \( X \).
(c) When someone dials this telephone number, what is the probability to receive a busy signal?

**Solution.** We have \( \lambda_a = 1/10 \min^{-1} = 0.1 \min^{-1}, \lambda_s = 1/5 \min^{-1} = 0.2 \min^{-1}, \Delta = 1 \min, \) and \( C = 1 \).

(a) Calculate probabilities,
\[
p_a = \lambda_a\Delta = (0.1)(1) = 0.1, \quad p_s = \lambda_s\Delta = (0.2)(1) = 0.2
\]
Possible states of \( X \) are 0 and 1 because the capacity is \( C = 1 \). Then
\[
p_{00} = 1-p_a = 0.9, \quad p_{01} = p_a = 0.1, \quad p_{10} = (1-p_a)p_s = (0.9)(0.2) = 0.18, \quad p_{11} = 1-0.18 = 0.82.
\]
The transition probability matrix is
\[
P = \begin{pmatrix} 0.9 & 0.1 \\ 0.18 & 0.82 \end{pmatrix}
\]

(b) Solve the system \( \pi P = \pi \) along with the normalizing condition \( \pi_0 + \pi_1 = 1 \).
\[
\begin{pmatrix} \pi_0 \\ \pi_1 \end{pmatrix} \begin{pmatrix} 0.9 & 0.1 \\ 0.18 & 0.82 \end{pmatrix} = \begin{pmatrix} \pi_0 \\ \pi_1 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} 0.9\pi_0 + 0.18\pi_1 = \pi_0 \\ 0.1\pi_0 + 0.82\pi_1 = \pi_1 \end{cases} \quad \Rightarrow \quad \begin{cases} 0.1\pi_0 = 0.18\pi_1 \\ 0.1\pi_0 = 0.18\pi_1 \end{cases}
\]
Now express \( \pi_0 = 1.8\pi_1 \) and substitute into the normalizing equation,
\[
\pi_0 + \pi_1 = 1 \quad \Rightarrow \quad 1.8\pi_1 + \pi_1 = 1 \quad \Rightarrow \quad 2.8\pi_1 = 1,
\]
so that
\[
\pi_1 = 5/14 \text{ or } 0.3571; \quad \pi_0 = 9/14 \text{ or } 0.6429
\]

(c) \( P(\text{full}) = \pi_1 = \begin{pmatrix} \pi_1 = 5/14 \text{ or } 0.3571 \end{pmatrix} \)
5. Suppose that the ages (in years) of five randomly selected application developers of Facebook are as follows:

\[21, 19, 25, 20, 28\]

(a) Find the sample mean and the sample standard deviation of these data.

(b) Using your answers in (a) and assuming a Normal distribution of ages, compute a 95% confidence interval for the mean age of Facebook application developers.

Solution.

(a) For the given data,

\[\bar{X} = \frac{1}{5} \sum_{i=1}^{5} X_i = 22.6, \quad S^2 = \frac{1}{5-1} \sum_{i=1}^{5} (X_i - \bar{X})^2 = 14.3, \quad S = \sqrt{S^2} = 3.78.\]

(b) Since \( n \) is small and the data follow a normal distribution, we need to compute a \( t \) confidence interval. Using the formula, the 95% confidence interval for mean age of developers is:

\[\bar{X} \pm t_{0.025, 4} S / \sqrt{n} = 22.6 \pm 2.776(3.78) / \sqrt{5} = 22.6 \pm 4.7 = (17.9, 27.3).\]

Here the critical point \( t_{0.025, 4} = 2.776 \) is obtained from the \( t \)-table.
6. The number $X$ of inquiries to databases in a computer code is believed to follow a Uniform($\theta, 2\theta$) distribution, where $\theta$ is an unknown parameter.

Two computer codes are selected at random. One of them has 40 inquiries, and the other has 50 inquiries. Based on this sample ($X_1 = 40$, $X_2 = 50$), estimate $\theta$ by the method of moments or the method of maximum likelihood.

Notice that there is only 1 parameter to estimate.

**Solution.**

- **Method of moments.** We have $M_1 = \overline{X} = \frac{40 + 50}{2} = 45$ and $\mu_1 = E(X) = \frac{\theta + 2\theta}{2} = (3/2)\theta$ for the Uniform($\theta, 2\theta$) distribution.

  Solve $M_1 = \mu_1 \Rightarrow 45 = (3/2)\theta \Rightarrow \hat{\theta} = 30$

- **Method of maximum likelihood.** Uniform($\theta, 2\theta$) has density $f(x) = \frac{1}{2\theta - \theta} = \frac{1}{\theta}$ for $\theta \leq x \leq 2\theta$.

  Then, the joint density of $X_1, X_2$ is

  $f(X_1, X_2) = \begin{cases} 
  \frac{1}{\theta^2}, & \theta \leq X_1, X_2 \leq 2\theta \\
  0, & \text{otherwise}
  \end{cases}$

  This density is maximized by the smallest possible $\theta$ such that $\theta \leq X_1, X_2 \leq 2\theta$. That is,

  $\hat{\theta} = \min\{\theta : \theta \leq 40 \text{ and } 50 \leq 2\theta\} = \min\{\theta : \theta \leq 40 \text{ and } \theta \geq 25\} = 25$