Markov Chains

Markov chain = discrete-time, discrete-state Markov stochastic process

It is described by

1) initial state distribution

\[ P \{X(0) = x\} = P_0(x) \]

2) transition probabilities

\[ P \{X(n + 1) = j \mid X(n) = i\} = P_n(i \rightarrow j) \]

If \( P_n(i \rightarrow j) = p_{ij} \) is independent of \( n \), it is a stationary Markov chain
Markov chains

\[ p_{ij}(k) = P \{ X(n + k) = j \mid X(n) = i \} \]

Based on \[
\begin{pmatrix}
P_0(1) \\
P_0(2) \\
\vdots
\end{pmatrix}
\] and \[
\begin{pmatrix}
p_{11} & p_{12} & p_{13} & \cdots \\
p_{21} & p_{22} & p_{23} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

how to compute

\[ p_{ij}(k) \]

\[ P \{ X_n = x \} \]

\[ \lim_{n \to \infty} P \{ X_n = x \} \]
MATRICES

\[ A = \{ A_{ij} \mid i = 1, \ldots, r, j = 1, \ldots, c \} \]

\[ = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1c} \\
A_{21} & A_{22} & \cdots & A_{2c} \\
\vdots & \vdots & \ddots & \vdots \\
A_{r1} & A_{r2} & \cdots & A_{rc}
\end{pmatrix} \]

is a matrix with \( r \) rows and \( c \) columns

\( i = \) row number

\( j = \) column number
Multiplying a row by a column

\[ A = (A_1, \ldots, A_n) = \text{“1 by } n\text{” matrix} \]

\[ B = \begin{pmatrix} B_1 \\ \vdots \\ B_n \end{pmatrix} = \text{“} n \text{ by 1” matrix} \]

Then

\[ AB = (A_1, \ldots, A_n) \begin{pmatrix} B_1 \\ \vdots \\ B_n \end{pmatrix} = \sum_{i=1}^{n} A_i B_i \]

**Example:** 3 hrs 25 min 45 sec = 12345 sec

because

\[
\begin{pmatrix} 3 & 25 & 45 \end{pmatrix} \begin{pmatrix} 3600 \\ 60 \\ 1 \end{pmatrix} = 12345
\]
A product of matrices

If \( A = \text{“}\!k\text{ by }m\!\text{”}\) matrix
\( B = \text{“}\!m\text{ by }n\!\text{”}\) matrix

then \( C = AB = \text{“}\!k\text{ by }n\!\text{”}\) matrix where

\[
C_{ij} = \sum_{h=1}^{n} A_{ih}B_{hj} = \left(\text{\(i^{th}\) row of } A\right)\left(\text{\(j^{th}\) column of } B\right)
\]

Example

\[
\begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 9 & -3 \\ -3 & 1 \end{pmatrix} = \text{interesting answer}
\]
Markov chains: transition probability matrix

\[ P = \{ p_{ij} \} = \left\{ P(i \rightarrow j), \quad i = 1, \ldots, n \right\} \]

\[ = \begin{pmatrix}
  p_{11} & p_{12} & \cdots & p_{1n} \\
p_{21} & p_{22} & & \cdots & p_{2n} \\
  \vdots & & \ddots & \vdots & \vdots \\
p_{n1} & p_{n2} & \cdots & p_{nn}
\end{pmatrix} \]

\( k \)-step transition probability matrix

\[ P(k) = \left\{ p_{ij}(k) \right\} \]

\[ = \left\{ P(i \rightarrow j \text{ in } k \text{ steps }), \quad i = 1, \ldots, n \right\} \]
Two-step transition probability matrix

\[ p_{ij}(2) = P \{X(2) = j \mid X(0) = i\} \]

\[ = \sum_k P(i \to k)P(k \to j) \]

(Law of Total Probability)

\[ = (p_{i1}, \ldots, p_{in}) \begin{pmatrix} p_{1j} \\ \vdots \\ p_{nj} \end{pmatrix} \]

Hence,

\[ P(2) = P \cdot P = P^2 \]
Markov chains

$k$-step transition probability matrix

\[ P(k) = \underbrace{P \cdot P \cdot \ldots \cdot P}_{k \text{ times}} = P^k \]

which means

\[ p_{ij}(k) = P \{ X(k) = j \mid X(0) = i \} = \sum_s \sum_t \sum_u \ldots \sum_z P(i \to s)P(s \to t) \cdots P(z \to j) \]
**Markov chains**

### Distribution of $X(k)$

If $P_0 = (P_0(1), \ldots, P_0(n)) = \text{pmf of } X(0)$,

$$P_k = (P_k(1), \ldots, P_k(n)) = \text{pmf of } X(k),$$

$$P = \{p_{ij}\} = \text{transition probability matrix},$$

then by the Law of Total Probability,

$$P_k(j) = P \{X(k) = j\}$$

$$= \sum_i P \{X(0) = i\} P \{X(k) = j \mid X(0) = i\}$$

$$= \sum_i P_0(i) p_{ij}(k)$$

Hence

$$P_k = P_0 P^k$$
Steady-state (limiting) probabilities

After very many transitions, what is the distribution of $X(k)$?

That is, $\lim_{k \to \infty} P_0 P_k = ?$

A Markov chain is regular if

$$p_{ij}(k) > 0$$

for some $k$ and all $i, j$

Fact. For a regular Markov chain, there is a limit

$$\lim_{k \to \infty} P^k = \Pi,$$

with all $\Pi_{ij} > 0$ and all rows of $\Pi$ being equal.
Markov chains

Steady-state distribution

Thus,

\[
\Pi = \begin{pmatrix}
\pi_1 & \pi_2 & \cdots & \pi_n \\
\pi_1 & \pi_2 & \cdots & \pi_n \\
\vdots & \vdots & \ddots & \vdots \\
\pi_1 & \pi_2 & \cdots & \pi_n \\
\end{pmatrix}
\]

\[
\pi = (\pi_1, \pi_2, \ldots, \pi_n) = \text{steady-state distribution}
\]

Then \( \lim_{k} P \{ X(k) = x \} \) is found as

\[
P_k(x) = (P_0(1), \ldots, P_0(n)) \begin{pmatrix}
\pi_x \\
\pi_x \\
\vdots \\
\pi_x \\
\end{pmatrix} = \pi_x
\]

and it does not depend on the initial state!
Computing $\pi$

$\Pi = \lim_{k \to \infty} P^k$, therefore, $\Pi P = \Pi$

Computation

$\pi = (\pi_1, \ldots, \pi_n)$ solves the system

\[
\begin{cases}
\pi P = \pi \\
\sum_{i} \pi_i = 1
\end{cases}
\]

($\pi$ = eigenvector of $P$)