STOCHASTIC PROCESSES

Let \( t \in T = \text{time} \)
\( \omega \in S = \text{outcome, element of the sample space} \)

\( X(t, \omega) = \text{stochastic process} \)

Discrete \( T \) \( \Rightarrow \) discrete time process
Connected \( T \) \( \Rightarrow \) continuous time process

\( X \in \mathcal{X} = \text{states of the process} \)

Discrete \( \mathcal{X} \) \( \Rightarrow \) discrete state process, chain
Connected \( \mathcal{X} \) \( \Rightarrow \) continuous state process
Stochastic processes

For any \( t \) \( \Rightarrow \) \( X_t(\omega) = \) random variable
For any \( \omega \) \( \Rightarrow \) \( X_\omega(t) = \) function of \( t \)
\((\text{path, trajectory, realization})\)

Examples

Temperature

Stock value

Number of jobs in a queue

Number of internet connections

Football score

\textit{Poisson process}

\textit{Binomial process}

\textit{Brownian motion}
Markov processes

$X(t)$ is a **Markov process** if for any $t_1 < \ldots < t_n < t$,

$$P \{ X(t) \in A \mid X(t_1) = x_1, \ldots, X(t_n) = x_n \} = P \{ X(t) \in A \mid X(t_n) = x_n \}$$

That is,

$$P \{ \text{future} \mid \text{past, present} \} = P \{ \text{future} \mid \text{present} \}$$

Markov dependence:

"**Future depends on the past only through the present**"
Counting processes

They count events.

Therefore, $X \in \{0, 1, 2, 3, \ldots\}$ and $X(t)$ is non-decreasing.

Examples:

Binomial process

Poisson process
Counting processes

Example

\[ X(t) = \text{number of arrived emails by the time } t \]

\[ Y(t) = \text{number of attachments by the time } t \]

E-mails are transmitted at \( t = 8, 22, 30, 32, 35, 40, 41, 50, 52, \text{ and } 57 \text{ min.} \) Only 3 e-mails contained attachments. One attachment was sent at \( t = 8, \) five more at \( t = 35, \) and two more attachments at \( t = 50. \)
**Binomial process**

**Binomial process** $X(t)$ is the number of successes by the time $t$ in a sequence of independent Bernoulli trials.

This process is
- discrete-time
- discrete-space
- counting
- Markov

Discrete time - one frame per trial

**Random variables**

$X(t) =$ number of arrivals by the time $t$

$Y =$ number of frames between arrivals

$T =$ time between arrivals
Binomial process

$X(t)$

Parameters

\[ \lambda = \text{arrival rate (successes per minute)} \]
\[ \Delta = \text{duration of 1 frame (minutes per frame)} \]
\[ p = \text{prob. of success (successes per frame)} \]

\[ p = \lambda \Delta, \quad \lambda = p/\Delta, \quad \Delta = p/\lambda \]
**Distributions**

\[
X(t) = \text{Binomial}(n, p) \quad \text{EX}(t) = np \quad \text{Var}X(t) = np(1 - p)
\]

\[
Y = \text{Geometric}(p) \quad \text{EY} = \frac{1}{p} \quad \text{Var}Y = \frac{1 - p}{p^2}
\]

\[
T = Y \Delta \quad \text{ET} = \frac{\Delta}{p} = \frac{1}{\lambda} \quad \text{Var}Y = \Delta^2 \frac{1 - p}{p^2}
\]

\[n = t/\Delta \quad \text{is the number of frames during time } t\]
Poisson process

\( X(t) = \) continuous-time process counting “rare events”, with properties:

- \( P \{ X(t + h) - X(t) = 1 \} \)
  \[ = P \{ 1 \text{ event in } [t,t+h] \} \]
  \[ = \lambda h + o(h), \quad \text{as} \quad h \to 0 \]

- \( P \{ X(t + h) - X(t) > 1 \} \)
  \[ = P \{ \text{more than 1 event in } [t,t+h] \} \]
  \[ = o(h), \quad \text{as} \quad h \to 0 \]

- For \( t_1 < t_2 < t_3 < t_4 \), the increments \( X(t_2) - X(t_1) \) and \( X(t_4) - X(t_3) \) are \textbf{independent}
Poisson process is continuous-time, discrete-state, Markov.

It is stationary if $\lambda$ is constant.

where $\lambda = E(\# \text{ events/min})$
Distributions

Let \( X(t) = \text{Binomial process} \)
\( = \) number of events during time \( t \)
\( = \) number of events during \( n = t/\Delta \) frames

\[
X(t) = \text{Binomial} \left( n = \frac{t}{\Delta}, p \right) \rightarrow \text{Poisson}(\lambda t)
\]

as \( \Delta \rightarrow 0, \)

hence \( p \rightarrow 0, \) \( n = t/\Delta \rightarrow \infty, \) \( np = tp/\Delta = \lambda t \)

\[
\mathbb{E}X(t) = \lambda t, \quad \text{Var}X(t) = \lambda t
\]
Interarrival times

Interarrival time = min \{t, X(t) \geq 1\}

<table>
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<th>Counting process</th>
<th>Interarrival times</th>
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<tr>
<td>Binomial, Poisson</td>
<td>$\Delta \cdot \text{Geometric}(p)$, Exponential($\lambda$)</td>
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Let $T_1, T_2, \ldots = $ successive interarrival times

$$P \{T_1 + \ldots + T_n > t\} = P \{X(t) < n\}$$

\[\uparrow\]

$\text{Gamma}(n, \lambda)$

\[\downarrow\]

$P \{T_1 + \ldots + T_n \leq t\} = P \{X(t) \geq n\}$

(Gamma-Poisson formula)