
- Read Sections 6.1-6.4
- Solve the practice problems below.
- Open Homework Assignment #7 and solve the problems.

1. Let $X_1, X_2, X_3, \ldots$ be a sequence of independent Gamma random variables. Define the partial sums $S_n = X_1 + \ldots + X_n$.
   
   a) Is $S_n$ a continuous time or discrete time process?
   b) Is $S_n$ a continuous state or discrete state process?
   c) Is $S_n$ a Markov process?
   d) Is $S_n$ a counting process?

   Explain your answers.
   
   a) Discrete time process because $n = 1, 2, 3, \ldots$
   b) Continuous state process because $S_n$, being a sum of continuous random variables, is a continuous random variable, which can take any positive value.
   c) A Markov process because the distribution of $S_{n+1} = S_n + X_{n+1}$ depends on $S_1, \ldots, S_n$ only through $S_n$.
   d) Not a counting process because it is a continuous state process - it cannot count events.

2. Let $X(t)$ denote the actual temperature at time $t$ during a day.
   
   a) Is $X(t)$ a continuous time or discrete time process?
   b) Is $X(t)$ a continuous state or discrete state process?
   c) Is $X(t)$ a Markov process?
   d) Is $X(t)$ a counting process?

   Explain your answers.
a) Continuous time process because $t$ can be any time during the day.

b) Continuous state process because temperature is a continuous variable.

c) Not a Markov process because given the temperature recordings at time $t$ and at earlier times during the day, the temperature an hour later does not solely depend on the temperature at time $t$.

d) Not a counting process since the states are not non-negative integers.

3. For a Binomial counting process with 2-second frames and the arrival rate of 10 arrivals per hour, calculate the probability of at least three new arrivals during an interval of 15 minutes.

We have $\Delta = 2 \text{ sec} = 1/30 \text{ min}$, $\lambda = 10 \text{ hrs}^{-1} = 1/6 \text{ min}^{-1}$, and therefore, $p = \lambda\Delta = (1/30)(1/6) = 1/180$.

During a 15-min interval, there are $n = t/\Delta = 15/(1/30) = 450$ frames. Large $n$ and small $p \Rightarrow$ use Poisson approximation. The number of arrivals during 450 frames is

$$Binomial(n = 450, p = 1/180) \approx Poisson(np = 2.5)$$

Then, from the Poisson table,

$$P[X(15) \geq 3] = 1 - F(2) = 1 - 0.544 = 0.456.$$  

4. Suppose that a number of defects coming from an assembly line can be modeled as a Binomial counting process with frames of one-half-minute length and probability $p = 0.02$ of a defect during each frame.

a) Find the probability of going more than 20 minutes without a defect.

b) Determine the arrival rate in units of defects per hour.

c) If the process is stopped for inspection each time a defect is found, on average how long will the process run until it is stopped?

Given:

$X_n$ is the number of defects in $n$ frames, it is Binomial $(n, p = 0.02)$;
$Y$ is the number of frames between two defects, it is Geometric $(p = 0.02)$;
$\Delta$ is the frame size = 0.5 minutes;
$T$ is the time between two successive defects = 0.5 $\cdot$ $Y$ min.
a) \( P(T > 20) = P(Y > 40) = P(X_{40} = 0) = (1 - 0.02)^{40} = 0.446. \)
b) Since 1 hour has 120 frames, \( \lambda = \text{no. of defects per hour} = np = 120(0.02) = 2.4 \) defects per hour.
c) \( E(T) = 0.5E(Y) = 0.5/0.02 = 25 \) minutes (because \( E(Y) = 1/p \)).

5. Customers of a certain internet service provider connect to the internet at the average rate of 3 customers per minute. Assuming Binomial counting process with 5-second frames,

a) Compute the probability of more than 10 new connections during the next 3 minutes.
b) Compute the mean and the standard deviation of the number of seconds between connections.

We have \( \lambda = 3 \text{ min}^{-1} \) and \( \Delta = 5 \text{ sec} = 1/12 \text{ min} \). Then \( p = \lambda\Delta = 0.25 \) is the probability of a new connection during a given frame.

a) During 3 minutes, there are \( 3/(1/12) = 36 \) frames. Large \( n; p \) is not small \(\Rightarrow\) use Normal approximation. The number of new connections \( X \) during 36 frames has Binomial \( (n = 36, p = 0.25) \) distribution that is approximately Normal with \( \mu = np = 9 \) and \( \sigma = \sqrt{np(1 - p)} = 2.60 \). Then

\[
P\{X > 10\} = P\{X > 10.5\} = P\{Z > 0.58\} = 1 - 0.72 = 0.28,
\]

from the table of Normal distribution.

b) For the time \( T \) between connections,

\[
E(T) = 1/\lambda = (1/3) \text{ min} = 20 \text{ sec}
\]

and

\[
SD(T) = \Delta \sqrt{\frac{1 - p}{p}} = (5 \text{ sec}) \sqrt{\frac{0.75}{0.25}} = 17.32 \text{ sec}
\]

6. Customers come to a self-service gas station at the rate of 20 per hour. Their arrivals are modeled by a Binomial counting process.
a) How many frames per hour should we choose, and what should be the length of each frame if the probability of an arrival during each frame is to be 0.05?

b) With this frames, find the expected value and standard deviation of the time between arrivals at the gas station.

Arrival rate \( \lambda = 20 \text{ hr}^{-1} \).

a) We want \( p = 0.05 \). Thus, \( \Delta = \text{duration of 1 frame} = p/\lambda = 0.05/(20 \text{ hrs}^{-1}) = (1/400) \text{ hrs} = 9 \text{ sec} \). Also, \( n = \# \text{ frames in 1 hr} = 1 \text{ hr}/\Delta = 400 \text{ frames} \).

b) Let \( T = \text{inter-arrival time} \). \( E(T) = \Delta/p = 9/0.05 = 180 \text{ sec} \) or 3 min. \( SD(T) = (\Delta/p)\sqrt{1-p} = 175.44 \text{ sec} \) or 2.92 min.

7. Repeat the last exercise using the Poisson process instead of the Binomial process. How do you explain the difference in standard deviations in this exercise and the previous exercise?

Assuming a Poisson model, \( T = \text{inter-arrival time} \sim \text{Exponential} (\lambda = 20 \text{ hr}^{-1}) \). We have:

\[
E(T) = SD(T) = 1/\lambda = 1/20 \text{ hrs} = 3 \text{ min}
\]

The expected inter-arrival time is the same as in the previous exercise, but the standard deviation is a little higher. The Binomial model is more restrictive; it does not allow more than 1 arrival during 1 frame (1 min) whereas the Poisson model allows any number of arrivals during any period of time. This is the reason the Poisson model gives higher variability to the time between arrivals.

8. The number of baseball games rained out in Mudville is a Poisson process with the arrival rate of 5 per 30 days.

a) Find the probability that there are more than 5 rained-out games in 15 days.

b) Find the probability that there are no rained-out games in seven days.

Given:

Arrival rate \( \lambda = 5 \text{ per 30 days} = (1/6) \text{ per day} \).

\( N(t) = \# \text{ arrivals (i.e., rained out games) in } t \text{ days}, \text{it is Poisson}(t/6) \).

a) \( P[X(15) > 5] = 1 - P[X(15) \leq 5] = 0.042 \) (using the Poisson(15/6 = 2.5) PMF)
b) \( P[N(7) = 0] = e^{-7/6} = 0.311 \)

9. The number of times a piece of military hardware must be serviced in an arctic environment is a Poisson process with an average of one service every 200 hours of operation.

a) If a mission involving the use of this hardware takes 24 hours to complete, what is the probability that it will be completed without service being required on the hardware?

b) If the probability is 0.95 that no service is required during a mission, what is the mission time?

Let \( X(t) \) = \# times service is required during \( t \) hours. It is Poisson(\( \lambda t \)), where \( \lambda = 1/200 \) hr\(^{-1} \).

Let \( T \) = time of first service. It is Exponential(\( \lambda = 1/200 \) hr\(^{-1} \))

a) \( P(T > 24) = e^{-24/200} = 0.887 \)

b) We need to find such \( t \) that \( P(T > t) = e^{-t/200} = 0.95 \). Solve this equation for \( t \) and get \( t = -200 \ln(0.95) = 10.26 \) hrs.

10. Shipments of paper arrive at a printing shop according to a Poisson process at a rate of 0.5 shipments per day.

a) Find the probability that the printing shop receives more than two shipments in a day.

b) If there are more than 4 days between shipments, all the paper will be used up and the presses will be idle. What is the probability that this will happen?

Given:

Arrival rate \( \lambda = 0.5 \) per day.

\( X(t) \) = number of arrivals (shipments) in \( t \) days, it is Poisson (0.5\( t \)),

\( T \) = inter-arrival time measured in days, it is Exponential (0.5).

a) \( P[X(1) > 2] = 1 - P[N(1) \leq 2] = 0.014 \) [using Poisson(0.5) PMF]

b) \( P[T > 4] = e^{-4(0.5)} = 0.135 \)