Ex. One of $N$ passwords opens a file. A user tries them in a random order, until the file opens. Find the distribution (PMF) of $X =$ the number of attempts.

Solution:
(1) Find all possible values of $X$.
(2) Compute the probability of each value.

Case I: Sampling without replacement.
(1) $X = 1, 2, \ldots, N$
(2) $P(1) = P(X=1) = \frac{1}{N}$

$P(2) = P(X=2) = P(1st \ fails \cap 2nd \ ok)$

$= P(1st \ fails) \cdot P(2nd \ ok \mid 1st \ fails)$
\[
= \frac{N-1}{N} \cdot \frac{1}{N-1} = \frac{1}{N}
\]

\[
P(3) = \frac{N-1}{N} \cdot \frac{N-2}{N-1} \cdot \frac{1}{N-2} = \frac{1}{N}
\]

\[
P(x) = \frac{1}{N} \quad \text{for all } x = 1, \ldots, N.
\]

Discrete Uniform

Case 2: Sampling with replacement

Every time, pick a random password out of \( N \) possible ones.

Values of \( X \): 1, 2, \ldots

\[
P(X = x) = P(\text{need } x \text{ attempts})
\]

\[
= P(\text{the first } x-1 \text{ attempts failed and attempt } x \text{ worked})
\]
\[(1 - \frac{1}{N})^{x-1}(\frac{1}{N}), \quad \text{for } x \geq 1.\]

\[
\sum_{x=1}^{\infty} P(x) = \sum_{x=1}^{\infty} (1 - \frac{1}{N})^{x-1}(\frac{1}{N})
\]

Geometric series

\[
\frac{(1 - \frac{1}{N})^{1-1} \frac{1}{N}}{1 - (1 - \frac{1}{N})} = 1
\]

It means that eventually, the right password will be found with probability 1.

Geometric Distribution
\[ P_X(x) = P(X = x) = \sum_y P(x, y) \]

\[ = \sum_y P(X = x \land Y = y) = \sum_y P(x, y) \]

\( Y, X = \text{random variable} \)

\( x, y = \text{values} \)
\[ \begin{array}{c|cccc|c} & P(x,y) & 0 & 1 & 2 & 3 & P_y(4) \\ \hline \hline Y & 0 & 0.2 & 0.1 & 0.1 & 0.1 & 0.5 \\ & 1 & 0.2 & 0.2 & 0.05 & 0.05 & 0.5 \\ \hline P_x(x) & 0.4 & 0.3 & 0.15 & 0.15 & 1 \\ \end{array} \]

Are \( X, Y \) independent?

Verify \( P(x,y) = P_x(x)P_y(y) \)

\[ P(0,0) = P_x(0)P_y(0) \]

\[ 0.1 = (0.4)(0.5) \quad \checkmark \]

\[ P(1,0) = P_x(1)P_y(0) \]

\[ 0.1 \neq (0.3)(0.5) \quad \text{← counter example.} \]

\( X, Y \) are dependent
Ex. $X, Y$ are independent, each with pmf

$P(0) = 0.4$
$P(1) = 0.3$
$P(2) = 0.3$

Find the pmf of $Z = X + Y$.

$P(x, y) = P_X(x) P_Y(y)$

\[
\begin{array}{c|ccc}
\hline
y & 0 & 1 & 2 \\
\hline
0 & .16 & .12 & .12 \\
1 & .12 & .09 & .09 \\
2 & .12 & .09 & .09 \\
\hline
\end{array}
\]

$P_Z(z)$

\[
\begin{array}{c|c}
\hline
z & P_Z(z) \\
\hline
0 & .16 \\
1 & .24 \\
2 & .33 \\
3 & .18 \\
4 & .09 \\
\hline
\end{array}
\]

\[\text{Answer:} \quad W = X - Y, \quad V = XY\]