M/M/1 queuing system

\[ A = \text{small, } A^2 = \text{negligible} \]

\[ P_A = \lambda_A A \]

\[ P_s = \lambda_s \Delta \]

Transition probabilities

\[ P_{00} = 1 - P_A = 1 - \lambda_A A \]

\[ P_{01} = P_A = \lambda_A A \]

For \( i \geq 1, \)

\[ P_{10} = P_s (1 - P_A) = \lambda_s \Delta (1 - \lambda_A A) \]

\[ = \lambda_s \Delta - \lambda_A \lambda_s A^2 \approx \lambda_s \Delta \]

\[ P_{11} = \lambda_A A \cdot \lambda_s \Delta + (1 - \lambda_A A) (1 - \lambda_s A) \approx 1 - \lambda_A A - \lambda_s \Delta \]

\[ P_{12} = P_A (1 - P_s) = \lambda_A A (1 - \lambda_s \Delta) \approx \lambda_A A, \text{ similarly for } \forall i \geq 1 \]
Transition probability matrix

\[ P = \begin{pmatrix}
1 - \lambda_A & \lambda_A & 0 & 0 & \cdots \\
\lambda_A & 1 - \lambda_A - \lambda_S & \lambda_A & 0 & \cdots \\
0 & \lambda_S & 1 - \lambda_A & \lambda_A & \cdots \\
0 & 0 & \lambda_S & 1 - \lambda_A & \lambda_A & \cdots \\
0 & 0 & 0 & \lambda_S & 1 - \lambda_A & \lambda_A & \cdots \\
\end{pmatrix} \]

Find its steady-state distribution.

1. \( (1 - \lambda_A) \pi_0 + \lambda_S \Delta \pi_1 = \pi_0 \)

\[ \overline{\pi_0} - \lambda_A \Delta \pi_0 + \lambda_S \Delta \pi_1 = \pi_0 \]

\[ - \lambda_A \pi_0 + \lambda_S \pi_1 = 0 \]

1st balance equation:

\[ \lambda_A \pi_0 = \lambda_S \pi_1 \]
\[ \begin{align*}
2 & \quad \lambda_A \Pi_0 + (1 - \lambda_A - \lambda_S \Delta) \Pi_1 + \lambda_S \Delta \Pi_2 = \Pi_1 \\
    \lambda_A \Pi_0 + \Pi_1 - \lambda_A \Pi_1 - \lambda_S \Delta \Pi_1 + \lambda_S \Delta \Pi_2 = \Pi_1 \\
    \lambda_A \Pi_0 - \lambda_A \Pi_1 - \lambda_S \Pi_1 + \lambda_S \Pi_2 = 0
\end{align*} \]

2nd balance equation:
\[ \lambda_A \Pi_1 = \lambda_S \Pi_2 \]

\[ \begin{align*}
3 & \quad \lambda_A \Pi_1 + (1 - \lambda_A - \lambda_S \Delta) \Pi_2 + \lambda_S \Delta \Pi_3 = \Pi_2 \\
    \lambda_A \Pi_1 + \Pi_2 - \lambda_A \Pi_2 - \lambda_S \Delta \Pi_2 + \lambda_S \Delta \Pi_3 = \Pi_2
\end{align*} \]

\[ \Rightarrow 3rd \ balance \ eqn: \ \lambda_A \Pi_2 = \lambda_S \Pi_3. \]

And so on...

\[ \lambda_A \Pi_x = \lambda_S \Pi_{x+1} \]

for \ \( x = 0, 1, 2, \ldots \)
Introduce \( r = \frac{\lambda A}{\lambda S} \)

= arrival-to-service ratio

= utilization

\[
\begin{align*}
\Pi_{x+1} &= r \Pi_x
\end{align*}
\]

Divide by \( S \)

Now,

\[
\Pi_x = r \Pi_{x-1} = r \cdot r \Pi_{x-2}
= \ldots = r^x \Pi_0.
\]

Normalizing condition: \( \sum_0^\infty \Pi_x = 1 \)

\[
\sum_0^\infty r^x \Pi_0 = \frac{\Pi_0}{1-r} = 1
\]

\[
\Rightarrow \quad \Pi_0 = \frac{1}{1-r}
\]

\[
\Pi_x = r^x \Pi_0 = \frac{r^x}{1-r}
\]
The steady-state distribution
is \( \Pi_x = P(X = x) = P(\text{the system}) \)
\[ = \Gamma^x (1 - \Gamma), \quad x = 0, 1, 2, \ldots \]

This is a shifted geometric distribution. That is,
\[ y = x + 1 \]
is geometric with \( p = 1 - \Gamma \).

We know \( E(y) = \frac{1}{1 - \Gamma} \)
\[ \text{Var}(y) = \frac{1 - p}{p^2} = \frac{\Gamma}{(1 - \Gamma)^2} \]

But \( x = y - 1 \).

So,
\[ E(X) = E(y) - 1 \]
\[ = \frac{1}{1 - \Gamma} - 1 = \frac{\Gamma}{1 - \Gamma} \]
\[ \text{Var}(X) = \text{Var}(y) = \frac{\Gamma}{(1 - \Gamma)^2} \]
Performance Characteristics

The expected number of jobs is

\[ E(X) = \frac{\Gamma}{1 - \Gamma} \]

where \( \Gamma = \frac{\lambda_A}{\lambda_S} \).

The number of jobs getting service is \( X_S \sim \text{Bernoulli} (\Gamma) \) because \( \Gamma = P(X > 0) = P(X_S = 1) \)

\[ = 1 - \pi_0 \]

\[ \implies EX_S = \Gamma \]

The expected number of waiting jobs is

\[ EX_W = EX - EX_S \]

\[ = \frac{\Gamma}{1 - \Gamma} - \Gamma = \frac{\Gamma - \Gamma + \Gamma^2}{1 - \Gamma} \]

\[ EX_W = \frac{\Gamma^2}{1 - \Gamma} \]
\[ X = X_w + X_s \]

\( X \) = total 
\( X_w \) = waiting 
\( X_s \) = served 

\[ \Pi_0 = P(X = 0) = P(\text{server is idle}) \]
\[ = P(W = 0) \]

\( W = \) waiting time

\[ \Gamma = 1 - \Pi_0 = \text{utilization} \]
\[ = P(X > 0) = P(\text{server is busy}) \]
\[ = P(W > 0) \]
Expected waiting time

At any time, there are \( X \) jobs in the system. When a new job arrives, its waiting time is

\[
W = \sum_{i=1}^{X-1} \text{(service times of the earlier jobs)} + \text{(remaining service time of the currently served job)}
\]

\[
E(W) = E(X-1) \mu_s + \mu_s
\]

\[
E(W) = E(X) \mu_s \quad \text{because of the loss of memory of Exp. service time.}
\]

\[
E(W) = \frac{\Gamma \mu_s}{1-\Gamma}
\]
Response Time

$= \text{total time spent in the system, from arrival to departure.}$

\[ T = W + S \]

response time \quad \text{waiting time} \quad \text{service time}

\[ ET = EW + ES \]

\[ = \frac{\mu s}{1 - \rho} + \mu s (1 - \rho) \]

\[ ET = \frac{\mu s}{1 - \rho} \]
Ex. M/M/1 system;
jobs arrive at the rate of
3 jobs per minute;
the average service time is 16 sec.
Compute performance characteristics.
\[ \lambda_A = 3 \text{ min}^{-1} = \frac{1}{20} \text{ sec}^{-1} \]
\[ \lambda_s = \frac{1}{16} \text{ sec} = \frac{1}{16} \text{ sec}^{-1} \]
\[ \Gamma = \frac{\lambda_A}{\lambda_s} = \frac{\frac{1}{20}}{\frac{1}{16}} = 0.8 < 1 \]
\[ \Pi_0 = 1 - \Gamma = 0.2 = P(\text{idle}) = P(W = 0) \]

W has a p.m.f. at 0
but it's continuous,
has a density for W > 0
W has a mixed distribution,
Exp. waiting time
\[ EW = \frac{\mu s}{1 - \gamma} = \frac{0.8 \cdot 16 \text{ sec}}{1 - 0.8} = 64 \text{ sec} \]

Exp. response time
\[ ET = \frac{s}{1 - \gamma} = \frac{16 \text{ sec}}{0.2} = 80 \text{ sec} \]

Exp. # jobs in the system
\[ EX = \frac{\gamma}{1 - \gamma} = \frac{0.8}{0.2} = 4 \text{ jobs} \]

Exp. # of waiting jobs
\[ EW = \frac{\gamma^2}{1 - \gamma} = \frac{0.8^2}{0.2} = 3.2 \text{ jobs} \]

The probability of more than 5 jobs waiting
\[ P(X_W > 5) = P(X > 6) = \sum_{x=7}^{8} \gamma^x (1 - \gamma) = \frac{\gamma^7 (1 - \gamma)}{1 - \gamma} = (0.8)^7 \]