Anti-Quiz # 8

Cars passing through a toll area and paying their tolls represent a *Bernoulli single-server queuing system with unlimited capacity*. On the average, a car arrives every 2 minutes, and it takes 1 minute to pay the toll. At 5:00 pm, there will be no cars in the toll area. Assuming 30-second frames, find the distribution of the number of cars in the toll area at 5:01 pm.

**Solution.**

This problem is similar to Practice problems #3 and #5.

We are given $\Delta = 1/2$ min, $\lambda_A = 1/2$ min$^{-1}$, and $\lambda_S = 1$ min$^{-1}$.

Then $p_A = \lambda_A \Delta = 1/4$ and $p_S = \lambda_S \Delta = 1/2$, and the transition probabilities are

\[
\begin{align*}
p_{00} &= 1 - p_A = 3/4 \\
p_{01} &= p_A = 1/4 \\
\text{and for } k \geq 1, \quad p_{k,k-1} &= p_S(1 - p_A) = (1/2)(3/4) = 3/8 \\
p_{k,k} &= p_A p_S + (1 - p_A)(1 - p_S) = (1/4)(1/2) + (3/4)(1/2) = 1/2 \\
p_{k,k+1} &= p_A(1 - p_S) = (1/4)(1/2) = 1/8.
\end{align*}
\]

There are two 30-second frames between 5:00 and 5:01. With 0 cars at 5:00, the number of cars after two frames can be 0, 1, or 2 with probabilities

\[
\begin{align*}
P(0) &= p_{00} p_{00} + p_{01} p_{10} = (3/4)^2 + (1/4)(3/8) = 9/16 + 3/32 = 21/32 \\
P(1) &= p_{00} p_{01} + p_{01} p_{11} = (3/4)(1/4) + (1/4)(1/2) = 3/16 + 1/8 = 5/16 \\
P(2) &= p_{01} p_{12} = (1/4)(1/8) = 1/32.
\end{align*}
\]

(It’s good to check that $P(0) + P(1) + P(2) = 21/32 + 10/32 + 1/32 = 1$.)

A photo printer can be working on one job and store up to 5 jobs in its memory. Its performance is modeled by a *Bernoulli single-server queuing process with limited capacity and 2-minute frames*. Jobs are sent to the photo printer at the average rate of one job every 6 minutes, and it takes 4 minutes to print the average job. At 10:00 am, the printer is busy, and there are no other jobs stored in its queue.

Compute the expected number of jobs in the system at 10:02 am, including the jobs being printed and stored, if any.

(Notice: this is not an M/M/1 system.)

**Solution.**

This problem is similar to Practice problem #3b.

We are given $\Delta = 2$ min, $\lambda_A = 1/6$ min$^{-1}$, and $\lambda_S = 1/\mu_S = 1/4$ min$^{-1}$.

Then $p_A = \lambda_A \Delta = (1/6)(2) = 1/3$ and $p_S = \lambda_S \Delta = (1/4)(2) = 1/2$.

There is 1 job in the system at 10:00, and it is just one frame between 10:00 and 10:02. Then, after 1 frame, the number of jobs can be 0, 1, or 2 with probabilities

\[
\begin{align*}
P(0) &= p_{10} = p_S(1 - p_A) = (1/2)(2/3) = 1/3 \\
P(1) &= p_{11} = p_A p_S + (1 - p_A)(1 - p_S) = (1/3)(1/2) + (2/3)(1/2) = 1/6 + 2/6 = 1/2 \\
P(2) &= p_{12} = p_A(1 - p_S) = (1/3)(1/2) = 1/6.
\end{align*}
\]

At 10:02, the expected number of jobs is

\[
E(X) = \sum_x xP(x) = (0)(1/3) + (1)(1/2) + (2)(1/6) = 1/2 + 1/3 + 5/6 = \boxed{5/6 \text{ jobs}}
\]