Capacity Allocation

Outline

◆ Two-class allocation
◆ Multiple-class allocation
◆ Demand dependence
◆ Bid prices
Objectives

- Profit maximization for all firms
- Revenue maximization for firms with fixed (sunk) costs
- Cost minimization for firms with fixed revenues
  - Supply chains with established, given prices
- Market share maximization for firms strategically sacrificing the current profit to achieve higher future periods
  - Amazon in the mid 2010s
  - American airlines response to the competitor on DFW ↔ FRA leg in the summer of 2018
- Asset utilization maximization for justifying investment into assets
  - A nonprofit hospital chain seeking state funding for a new building
Booking Limits for 2 Fare Classes

Only one decision variable: \( b = b_2 \)
- Second class has booking limit \( b \)
- The first class has protection level of \( C-b \)

Demands for
- Full-fare class \( d_f \) whose cumulative density is \( F_f \);
- Discount fare class \( d_d \) whose cumulative density is \( F_d \).

Prices for
- Full-fare class \( p_f \);
- Discount fare class \( p_d \).
2-class Problem
Increasing the Booking Limit

\[ F_d(b): \text{No second class is rejected with } b \text{ or } b+1 \]
Remaining capacity for the first class is the same
\[ d_d \leq b \] discount demand
too low to be limited

\[ b \text{ up by 1} \]

\[ d_d > b \]
\[ 1 - F_d(b): \text{Remaining capacity for the first class is less by 1} \]
\[ d_f \leq C-b \]
\[ F_f(C - b): \text{Discount fare class accepted because of } b \leftarrow b + 1 \]
\[ p_d - 0 > 0 \]

\[ d_f > C-b \]
\[ 1 - F_f(C - b): \text{Full-fare class rejected because of } b \leftarrow b + 1 \]
\[ p_d - p_f < 0 \]

Marginal effect on revenue

\[ 0 - 0 = 0 \]

\[ b \text{ keep constant} \]

Expected marginal effect on revenue

\[ = F_d(b)(0) + [1 - F_d(b)] \left[ [1 - F_f(C - b)](p_d - p_f) + F_f(C - b)p_d \right] \]
\[ = [1 - F_d(b)]\left[p_d - [1 - F_f(C - b)]p_f \right] \]
\[ = [1 - F_d(b)]p_f \left[F_f(C - b) - \left(1 - \frac{p_d}{p_f}\right)\right] \]
2 Fare Classes

Booking Limit and Protection Level

The critical term is $F_f(C - b) - \left(1 - \frac{p_d}{p_f}\right)$.

$F_f(C - b)$ decreases as $b$ increases because $F_f$ is a cumulative density and increases in its argument.

If $b = 0$ and $F_f(C - [b = 0]) \leq \left(1 - \frac{p_d}{p_f}\right)$, set $b = 0$.

In this case, the discount price is very low so it is not worth opening up the discount class.

For any $b$, increase $b$ as long as $F_f(C - b) > \left(1 - \frac{p_d}{p_f}\right)$.

For a continuous full fare demand distribution,

$$C - b^* = F_f^{-1}\left(1 - \frac{p_d}{p_f}\right)$$

and $b^* \leq C$.

$$b^* = \max\{C - F_f^{-1}\left(1 - \frac{p_d}{p_f}\right), 0\}$$

$$y^* = C - b^* = C - \max\{C - F_f^{-1}\left(1 - \frac{p_d}{p_f}\right), 0\}$$

$$= C + \min\{-C + F_f^{-1}\left(1 - \frac{p_d}{p_f}\right), 0\}$$

$$= \min\{F_f^{-1}\left(1 - \frac{p_d}{p_f}\right), C\}$$

Or $\frac{p_d}{p_f} = P(d_f > y^*)$.
2 Fare Classes
Protection Level with Uniform Demand

Suppose that the first class (full-fare) demand is uniform between 20 and 40 seats on a flight. Moreover, first class passengers pay $1000, while the discount fare is $300.
What are booking limits for discount fare if the plane has $C = 100$ or $C = 30$ seats?

Protection level $y$ first:

$$F_f(x) = \frac{x-20}{20} \text{ for } 20 < x < 40.$$  
$$F_f^{-1}(u) = 20 + 20u \text{ for } 0 < u < 1.$$  
$$F_f^{-1}\left(1 - \frac{p_d}{p_f}\right) = F_f^{-1}\left(1 - \frac{300}{1000}\right) = 34.$$  

If capacity=100, booking limit=100-34=66 for the discount class. If capacity=30, booking limit=\((30-34)^+\)=0 for the discount class.
2 Fare Classes
Protection Level with Normal Demand

Suppose that the first class (full-fare) class demand is normal with mean 30 seats and standard deviation 5. Moreover, first class passengers pay $1000, while the discount fare is $300.

What are booking limits for discount fare if the plane has $C = 100$ or $C = 30$ seats?

Standard normal cumulative distribution $\Phi(z)$.

$$\Phi_z^{-1}\left(\frac{7}{10}\right) = \text{norminv}(0.7, \text{mean } = 0, \text{stdev } = 1)$$

$$= 0.52 = z.$$ 

Protection level = $\text{mean } + z \times \text{stdev } = 30 + (0.52)5 = 32.6$.

Or use $\text{norminv}(0.7, \text{mean } = 30, \text{stdev } = 5) = 32.6$.

Let us round up the protection level to 33.

If capacity=100, booking limit=100-33=67 for the discount class.

If capacity=30, booking limit=(30-33)^+ = 0 for the discount class.
Multiple Fare Classes
Nested Protection Levels

- Fare classes indexed by $j=1,2, \ldots, n$. $j=1$ is the highest class, while $j=n$ is the lowest class.
- Protection level $y_i$ for class $i$ protects future reservations or classes $j=\{1,2, \ldots, i\}$.

Protection levels

<table>
<thead>
<tr>
<th>Periods</th>
<th>$n$</th>
<th>$n-1$</th>
<th>3</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Independent Demands</td>
<td>$d_n$</td>
<td>$d_{n-1}$</td>
<td>$d_3$</td>
<td>$d_2$</td>
<td>$d_1$</td>
</tr>
<tr>
<td>Monotone Prices</td>
<td>$p_n &lt; p_{n-1}$</td>
<td>$p_3 &lt; p_2 &lt; p_1$</td>
<td></td>
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<tr>
<td>Time-wise Separated Bookings</td>
<td>$x_n$</td>
<td>$x_{n-1}$</td>
<td>$x_3$</td>
<td>$x_2$</td>
<td>$x_1$</td>
</tr>
</tbody>
</table>
Multiple Fare Classes
Increasing the Booking Limit

- **$F_3(b_3)$**: No third class is rejected with $b_3$ or $b_3+1$
  - Remaining capacity for first/second class is the same

  - $d_3 \leq b_3$
    - $b_3$ up by 1
      - 1st class rejected because of $b_3 \leftarrow b_3 + 1$
        - $p_3 - p_1 < 0$
      - 2nd class rejected because of $b_3 \leftarrow b_3 + 1$
        - $p_3 - p_2 < 0$
      - Low 1st-2nd class demand
        - neither rejected
        - $p_3 - 0 > 0$

  - $d_3 > b_3$
    - $b_3$ keep constant
    - 1st class rejected because of $b_3 \leftarrow b_3 + 1$
      - $p_3 - p_1 < 0$

- **$1 - F_3(b_3)$**: Sell 1 more to 3rd class
  - Remaining capacity for 1st-2nd classes is less by 1

- **$b_3$ limits 3rd class to keep space for 1st-2nd classes**

**Challenge**: Computing the probability of 1st or 2nd class rejection by increasing the booking limit of the third class.

**Marginal effect on revenue**

- $0 - 0 = 0$
- $p_3 - p_1 < 0$
- $p_3 - p_2 < 0$
- $p_3 - 0 > 0$

- 0
Expected Marginal Seat Revenue (EMSR-a)
Version-a Heuristic: Protection Level Decomposition

- In period 3, when accepting bookings for class 3, protect class 1 and class 2.
- Let $y_{1,3}$ and $y_{2,3}$ be the protection levels of first and second classes against the third class.
- We can set
  
  $$y_{1,3} = F_1^{-1} \left(1 - \frac{p_3}{p_1}\right) \quad \text{and} \quad y_{2,3} = F_2^{-1} \left(1 - \frac{p_3}{p_2}\right)$$

- The protection level for both first and second classes while booking for class 3 is
  
  $$y_2 = y_{1,3} + y_{2,3} \quad \text{← Computation allotment-like but execution is still nested}$$

- In general,
  
  $$y_i = y_{1,i+1} + y_{2,i+1} + \ldots + y_{i,i+1} = \sum_{j=1}^{i} y_{j,i+1}$$

  where $y_{j,i+1}$ is the protection level of class $j$ from class $i+1$. That is,

  $$y_{j,i+1} = F_j^{-1} \left(1 - \frac{p_{i+1}}{p_j}\right) \quad \text{or} \quad \frac{p_{i+1}}{p_j} = P(d_j > y_{j,i+1})$$

- Protect class $j$ more against class $i+1$ if
  - Class $j$ demand is larger
  - Class $j$ price is larger
Expected Marginal Seat Revenue (EMSR-b)
Version-b Heuristic: Remaining Demand Aggregation

- In period 3, when accepting bookings for class 3, protect class 1 and class 2.
- Aggregate classes 1 and 2 into a new demand class \{1,2\}.
- Set demand and price for the class \{1,2\}:
  \[ d_{\{1,2\}} = d_1 + d_2 \quad \text{and} \quad p_{\{1,2\}} = \frac{p_1 E d_1 + p_2 E d_2}{E d_1 + E d_2} \]
- The protection level for the aggregate class \{1,2\} while booking for class 3 is
  \[ p_3 / p_{\{1,2\}} = P(d_{\{1,2\}} > y_2) \]

- In general,
  \[ d_{\{1,2,\ldots,i\}} = d_1 + d_2 + \ldots + d_i \quad \text{and} \quad p_{\{1,2,\ldots,i\}} = \sum_{j=1}^{i} p_j \frac{E d_j}{\sum_{k=1}^{i} E d_k} \]
  \[ \frac{p_{i+1}}{p_{\{1,2,\ldots,i\}}} = P(d_{\{1,2,\ldots,i\}} > y_i) \]

- Protect class \{1,2,\ldots,i\} more against class \(i+1\) if
  - Class \{1,2,\ldots,i\} demand is larger
  - Class \{1,2,\ldots,i\} price is larger
Version-b Heuristic with Normal Demands

- With normal independent demands mean $\mu_i$ and standard deviation $\delta_i$

$$
\mu_{\{1,2,\ldots,i\}} = \mu_1 + \mu_2 + \ldots + \mu_i, \quad \delta_{\{1,2,\ldots,i\}} = \sqrt{\sum_{j=1}^{i} \delta_j^2}
$$

and

$$
p_{\{1,2,\ldots,i\}} = \frac{\sum_{j=1}^{i} p_j d_j}{\mu_{\{1,2,\ldots,i\}}}.
$$

The protection level for the aggregate class $\{1,2,\ldots,i\}$ while booking for class $i+1$ is

- $y_{\{1,2,\ldots,i\}} = \text{Norminv} \left( 1 - \frac{p_{i+1}}{p_{\{1,2,\ldots,i\}}}, \mu_{\{1,2,\ldots,i\}}, \delta_{\{1,2,\ldots,i\}} \right)$

Example: Suppose that first and second class demands are normal with the same parameters

$\mu_1 = \mu_2 = 20; \quad \delta_1 = \delta_2 = 5$

What is the distribution of the demand of aggregate class $\{1,2\}$?

The aggregate distribution is also Normal with parameters:

$$
\mu_{\{1,2\}} = 40; \quad \delta_{\{1,2\}} = \sqrt{5^2 + 5^2} \approx 7.1
$$
Comparison of Version-a & Version-b Heuristics under Normal Demands

◆ Example: Suppose that first and second class demands are normal with the same parameters $\mu_1 = \mu_2 = 20; \ \delta_1 = \delta_2 = 5$.

Then the distribution of the demand of aggregate class \{1,2\} is also Normal with parameters:

$$\mu_{\{1,2\}} = 40; \ \delta_{\{1,2\}} = \sqrt{5^2 + 5^2} \approx 7.1$$

Furthermore, suppose that the prices for classes are $p_1=1000; p_2=700; p_3=400$. Find the protection level for classes 1 and 2 by using EMSR-a and EMSR-b heuristics.

EMSR-a heuristic:

◆ Protection level for class 1 from class 3:

$$y_{1,3} = norminv\left(1 - \frac{400}{1000}, 20, 5\right) = 21.27$$

◆ Protection level for class 2 from class 3:

$$y_{2,3} = norminv\left(1 - \frac{400}{700}, 20, 5\right) = 19.10$$

◆ Protection level for classes 1-2 from class 3:

$$y_2 = 21.27 + 19.10 = 40.37$$

EMSR-b heuristic:

◆ Price for the aggregate class \{1,2\}=850.

◆ Protection level for the aggregate class \{1,2\} while booking for class 3 is

$$y_{1,2} = norminv\left(1 - \frac{400}{850}, 40, 7.1\right) = 40.52$$

◆ Protection level for classes 1-2 from class 3:

$$y_2 = 40.52$$

The protection levels above are almost the same.
2-class Problem
Demand Dependence By Buying up

\[ F_d(b): \text{No second class is rejected with } b \text{ or } b+1 \]

\[ d_d \leq b \]
Remaining capacity for first class is the same

\[ 1-F_d(b) \]: Remaining capacity for first class is less by 1

\[ d_d > b \]

If \( p_d < \alpha p_f \), do not book for the 2nd class
Increasing \( b \), we lose the opportunity of selling at \( p_f \) to low-paying customers who buy up if their class is closed

\[ d_f \leq C-b \]

\[ 1-F_f(C-b) \]: First class rejected because of \( b+1 \)

\[ d_f > C-b \]

\[ 0-0=0 \]

\[ p_d - p_f < 0 \]

\[ p_d - 0 > 0 \]

\[ p_d - p_f < 0 \]

\[ 1-\alpha \]

\[ \alpha: \text{Rejected class 2 buys up class 1} \]

\[ F_f(C-b) \]: Low first class demand

\[ \alpha \]

\[ p_d - p_f < 0 \]

\[ p_d - 0 > 0 \]

\[ 0 \]

Marginal effect on revenue

Expected marginal effect on revenue

\[
= F_d(b)(0) + [1 - F_d(b)] \left[ [1 - F_f(C - b)](p_d - p_f) + F_f(C - b)(p_d - \alpha p_f) \right]
\]

\[
= [1 - F_d(b)][p_d - \alpha p_f - [1 - F_f(C - b)](1 - \alpha)p_f]
\]

\[
= [1 - F_d(b)](1 - \alpha)p_f \left[ F_f(C - b) - \left( 1 - \frac{p_d - \alpha p_f}{(1-\alpha)p_f} \right) \right]
\]

\[
= F_d(b)(0) + [1 - F_d(b)] \left[ [1 - F_f(C - b)](p_d - p_f) + F_f(C - b)(p_d - \alpha p_f) \right]
\]

\[
= [1 - F_d(b)][p_d - \alpha p_f - [1 - F_f(C - b)](1 - \alpha)p_f]
\]

\[
= [1 - F_d(b)](1 - \alpha)p_f \left[ F_f(C - b) - \left( 1 - \frac{p_d - \alpha p_f}{(1-\alpha)p_f} \right) \right]
\]
2 Fare Classes
Booking Limit and Protection Level

The critical term is \( F_f(C - b) - \left( \frac{p_f - p_d}{(1 - \alpha)p_f} \right) \).

\( F_f(C - b) \) decreases as \( b \) increases because \( F_f \) is a cumulative density and increases in its argument.

If \( b = 0 \) & \( F_f(C - [b = 0]) \leq \left( \frac{p_f - p_d}{(1 - \alpha)p_f} \right) \), set \( b = 0 \).

In this case, the discount price is very low so it is not worth opening up the discount class.

For any \( b \), increase \( b \) as long as \( F_f(C - b) > \left( \frac{p_f - p_d}{(1 - \alpha)p_f} \right) \).

For a continuous full fare demand distribution, \( C - b^* = F_f^{-1} \left( \frac{p_f - p_d}{(1 - \alpha)p_f} \right) \) and \( b^* \leq C \).

\[
\begin{align*}
  b^* &= \max \left\{ C - F_f^{-1} \left( \frac{p_f - p_d}{(1 - \alpha)p_f} \right) , 0 \right\} \\
  y^* &= \min \left\{ F_f^{-1} \left( \frac{p_f - p_d}{(1 - \alpha)p_f} \right) , C \right\} \\
  \text{Or} & \quad \frac{p_f - p_d}{(1 - \alpha)p_f} = P(d_f \leq y^*) \\
  \text{Or} & \quad \frac{p_d - \alpha p_f}{(1 - \alpha)p_f} = P(d_f > y^*)
\end{align*}
\]
2 Fare Classes
Protection Level with Uniform Demand

Suppose that the first class (full-fare) class demand is uniform between 20 and 40 seats on a flight. Moreover, first class passengers pay $1000, while the discount fare is $300. Buy up probability is 0.2. What are booking limits for discount fare if the plane has $C = 100$ or $C = 30$ seats?

Protection level $y$ first:

\[
F_f(x) = \frac{x - 20}{20} \quad \text{for} \quad 20 < x < 40.
\]

\[
F_f^{-1}(u) = 20 + 20u \quad \text{for} \quad 0 < u < 1.
\]

\[
F_f^{-1}\left(\frac{p_f - p_d}{(1 - \alpha)p_f}\right) = F_f^{-1}\left(\frac{1000 - 300}{(1 - 0.2)1000}\right) = 37.5.
\]

Let us round up the protection level to 38.

If capacity=100, booking limit=100-38=62 for the discount class. If capacity=30, booking limit=(30-38)+=0 for the discount class.

Booking limit with buy-up probability 0 is 67, it decreases to 62 with buy-up probability of 0.2.

\[
\frac{p_f - p_d}{(1 - \alpha)p_f} = \frac{7}{8}
\]
2 Fare Classes

Protection Level with Normal Demand

Suppose that the first class (full-fare) class demand is normal with mean 30 seats and standard deviation 5. Moreover, first class passengers pay $1000, while the discount fare is $300. Buy up probability is 0.2.

What are booking limits for discount fare if the plane has $C = 100$ or $C = 30$ seats?

Standard normal cumulative distribution $\Phi(z)$.

$$\Phi_z^{-1}\left(\frac{7}{8}\right) = \text{norminv}(7/8, \text{mean} = 0, \text{stdev} = 1) = 1.15 = z.$$  

Protection level $= \text{mean} + z \times \text{stdev} = 30 + (1.15)5 = 35.75$.

Or use $\text{norminv}(7/8, \text{mean} = 30, \text{stdev} = 5) = 35.75$.

Let us round up the protection level to 36.

If capacity=100, booking limit=100-36=64 for the discount class.
If capacity=30, booking limit=(30-36)$^+$=0 for the discount class.
**Bid (Displacement) Prices**

- **Bid price** for a flight is the threshold price that depends on the remaining capacity & time until departure, such that

  
  a booking request is \{\text{accepted if the fare} > \text{ more} \}, \text{rejected if the fare} < \text{ less} \} \text{ than the threshold price}

- **Bid price** = \((1 + \text{ Profit margin}) \ast (\text{Displacement cost})\), reminiscent of cost-plus pricing
  - To focus on the rest, assume profit margin = 0

- **Bid price (Displacement cost)** is the marginal (opportunity) cost of capacity.

- How much to charge to a request to allocate 1 unit capacity to that request?
  - Profit \(\Pi(b)\) with the capacity \(b\) versus the profit \(\Pi(b - 1)\) with the remaining capacity \(b - 1\) after allocation
    - **Accept** if the fare > \(\Pi(b) - \Pi(b - 1)\)
    - **Reject** if the fare < \(\Pi(b) - \Pi(b - 1)\)

- **Bid price (Displacement cost)** = \(\Pi(b) - \Pi(b - 1) = \frac{d}{d b} \Pi(b)\)

- Bid price is the marginal cost of capacity.
  - From linear demand & constrained capacity: The profit \(\Pi(p) = (p - c) (D - mp)\) and the capacity \(b\)
    - **When the capacity is not binding**, i.e., \(b \geq D - mp_0\) where \(p_0\) is the maximizer of \(\Pi(p)\),
      \[
      \frac{d}{d b} \Pi(b) = 0
      \]
    - **When the capacity is binding**,\[
      \frac{d}{d b} \Pi(b) = \frac{D}{m} - \frac{2b}{m} - c
      \]
Towards Bid Price Computation

- Ex: Suppose that $d(p) = 200 - 10p$ if $0 \leq p \leq 20$. Find total margin maximizing price when the cost is $c = 0$ and the capacity constraint is $b = 20$.
  - $b = 20$
    - From $\max_p pd(p) = 200p - 10p^2$, the unconstrained price $p_0 = 10$.
      - Binding because $d(10) = 100 > 20 = b$.
      - To sell all of the capacity $20 = b = d(p) = 200 - 10p$ or $p = 18$.
    - $\Pi(b = 20) = 360$.
  - After selling 1, $b = 19$
    - From $\max_p pd(p) = 200p - 10p^2$, the unconstrained price $p_0 = 10$.
      - Binding because $d(10) = 100 > 19 = b$.
      - To sell all of the capacity $19 = b = d(p) = 200 - 10p$ or $p = 18.1$.
    - $\Pi(b = 19) = 343.9$.
    - $\Pi(20) - \Pi(19) = 360 - 343.9 = 16.1$.
    - Charge 16.1 to compensate for the 20th unit.
  - This price is different from 18 because of discreteness, see below.
  - Suppose we set $p = 18$ slightly higher than 16.1.

- When capacity is binding, bid price $\approx$ optimal price, subject to discreteness of sellable capacity units
  - When capacity is not discrete, but continuous at small $\epsilon$, bid price is $\frac{\Pi(b) - \Pi(b-\epsilon)}{\epsilon}$, which is $\frac{\partial \Pi(b)}{\partial b}$.
    - When the capacity is binding, $\Pi(b) = bp$ and $p = d^{-1}(p)$ so $\frac{\partial \Pi(b)}{\partial b} = \frac{\partial}{\partial b} bp = p = d^{-1}(p)$.
  - In the discrete example above $\frac{\Pi(20) - \Pi(19)}{1} = 16.1$ while $d^{-1}(p) = 18$. 
Bid Prices over Time

- In a static setting, the bid price would be
  \[ \frac{d}{db} \Pi(b) \approx \Pi(b) - \Pi(b - 1) \]

- In the dynamic setting, the marginal cost of capacity when \( t \) units of time left is
  \[ \frac{d}{db} \Pi(b, t) \approx \Pi(b, t) - \Pi(b - 1, t) \]

- Bid price decreases with large capacity (supply) and with less amount of remaining time (demand).

- Opportunity cost of capacity depends on the capacity \( b \).
  - If \( b \) is very large, the opportunity cost is low: Not enough customers to sell the large capacity after denying the current.

- Opportunity cost of capacity also depends on the time \( t \) that remains to sell the capacity.
  - If \( t \) is very small, the opportunity cost is low: Not enough time to sell after denying the current.
Bid Price Path

In comparison to fare classes, booking by bid prices

- Is simpler as there is a single bid price for a single flight while there are many fare classes
- Is more complex as the bid prices must be computed dynamically while fare class booking limits can be static.

*No free lunch* by booking via fare classes or bid prices.
Bid Prices with Constant Prices

- When capacity is non-binding, bid price $= 0 < \text{optimal price}$
  - Value-based pricing: Optimal price
  - (Opportunity) Cost-based pricing: Bid price
    » Opportunity cost of one fewer unit of capacity is nothing if capacity > demand

- Bid price at time $t$ is the value of capacity $\Pi(b, t) - \Pi(b - 1, t)$

- A request acceptance/rejection formulation.
  - $\Pi(b, t)$ is optimal profit obtained by selling $b$ capacity to $h$ & $l$ type customers over $t, t - 1, \ldots, 1$.
  - $h$ and $l$ customers are high-type (full fare) paying $p_h$ and low-type (discount fare) paying $p_l$.
  - $q_h$ and $q_l$ are probability of $h$ and $l$ customers showing up, $q_h + q_l \leq 1$

<table>
<thead>
<tr>
<th>( \Pi(b, t) )</th>
<th>No customer request</th>
</tr>
</thead>
<tbody>
<tr>
<td>$= (1 - q_h - q_l) \Pi(b, t - 1)$ + $q_h \max { \Pi(b, t - 1), p_h + \Pi(b - 1, t - 1) }$ + $q_l \max { \Pi(b, t - 1), p_l + \Pi(b - 1, t - 1) }$</td>
<td>Capacity $\downarrow b - 1$ with an acceptance of $h$ request</td>
</tr>
</tbody>
</table>

| \( \Pi(0, t) \) | Capacity $\downarrow b - 1$ with an acceptance of $l$ request |
|------------------| Revenue with no capacity or no demand (time). |

\( \Pi(0, 0) = \Pi(b, 0) = 0 \)

This dynamic program can be evaluated in Excel for various $(p_h, p_l)$ & $(q_h, q_l)$; see bidPrices.xlsx.
Bid Prices with Optimized Prices

- Assume that future prices = the current price and solve: Exemplified on the previous pages

- In reality, future prices can respond to current demand, so they should be different from the current one
  - Future prices are anticipatory with respect to current demand

- To allow for future prices to respond, we need a recursive revenue computation.
  - \( \Pi(b, t) \) is optimal profit obtained by selling \( b \) capacity to \( h \) and \( l \) type customers.
  - \( h \) and \( l \) customers are high-type (full fare) and low-type (discount fare).
  - \( WTP_h \) and \( WTP_l \) are willingness to pay of \( h \) and \( l \) customers
  - \( q_h \) and \( q_l \) are probability of \( h \) and \( l \) customers showing up, \( q_h + q_l \leq 1 \)
  - \( p_h \) and \( p_l \) are current prices for \( h \) and \( l \) customer

\[
\Pi(b, t) = \max_{p_h, p_l} \left( q_h P(WTP_h > p_h) p_h + q_l P(WTP_l > p_l) p_l \right) \\
+ \left( q_h P(WTP_h > p_h) + q_l P(WTP_l > p_l) \right) \Pi(b - 1, t - 1) \\
+ \left( 1 - q_h P(WTP_h > p_h) - q_l P(WTP_l > p_l) \right) \Pi(b, t - 1)
\]

\[
\Pi(0, t) = \Pi(b, 0) = 0
\]
Prices over Time

- With \( t = 20 \) days until flight departure/room check-in/car pick-up, decide on the price:
  - \( d(p, t = 20) = 200 - 10p \) and \( b(t = 20) = 20 \) yields \( p(t) = 18 \).

- Demand for the remaining \( t \) days can be given as \( d(p, t) = 10t - 10p \).

- Suppose 1 person buys on day 20, so \( b(t = 19) = 19 \).

- On day 19, \( b(19) = 19 \) and \( d(p, 19) = 190 - 10p \).
  - For \( b(19) = 19 \),
    » max \( pd(p, 19) = 190p - 10p^2 \) \( \Rightarrow p_0 = 9.5 \). But \( d(9.5,19) = 95 > 19 \).
    » To sell all of the capacity \( 19 = 190 - 10p \) or \( p = 17.1 \). \( \Pi(b = 19, t = 19) = 19 \times 17.1 = 324.9 \).
  - For \( b(19) = 18 \),
    » From max \( pd(p) = 190p - 10p^2 \) \( \Rightarrow p_0 = 9.5 \). But \( d(9.5,19) = 95 > 18 \).
    » To sell all of the capacity \( 18 = 190 - 10p \) or \( p = 17.2 \). \( \Pi(b = 18, t = 19) = 309.6 \).
  - \( \Pi(19,19) - \Pi(18,19) = 324.9 - 309.6 = 15.3 \). Suppose we set \( p = 17 \) slightly higher than 15.3.
  - Suppose nobody buys at \( p = 17 \).

- On day 18, \( b(18) = 19 \) and \( d(p, 18) = 180 - 10p \).
  - For \( b(18) = 19 \),
    » From max \( pd(p) = 180p - 10p^2 \) \( \Rightarrow p_0 = 9 \). But \( d(9,18) = 90 > 19 = b(18) \).
    » To sell all of the capacity \( 19 = 180 - 10p \) or \( p = 16.1 \). \( \Pi(b = 19, t = 18) = 305.9 \).
  - For \( b(18) = 18 \),
    » From max \( pd(p) = 180p - 10p^2 \) \( \Rightarrow p_0 = 9 \). But \( d(9,18) = 90 > 18 = b(18) \).
    » To sell all of the capacity \( 18 = 180 - 10p \) or \( p = 16.2 \). \( \Pi(b = 18, t = 18) = 291.6 \).
  - \( \Pi(18,18) - \Pi(17,18) = 324.9 - 309.6 = 14.3 \). Suppose we set \( p = 16 \) slightly higher than 14.3.
  - Three buy at 16 on day 18, and so on …
Summary

- Two-class allocation
- Multiple-class allocation
- Demand dependence
- Bid prices

Based on Phillips (2005) Chapter 7
Normal Density Function

Excel statistical functions:
Density function (pdf) at \( x \) : \( \text{normdist}(x, \text{mean}, \text{st } _\text{dev}, 0) \)
Cumulative function (cdf) at \( x \) : \( \text{normdist}(x, \text{mean}, \text{st } _\text{dev}, 1) \)
Cumulative Normal Density

Excel statistical functions:
Cumulative function (cdf) at $x$: $\text{normdist}(x, mean, st\_dev, 1)$
Inverse function of cdf at "prob": $\text{norminv}(\text{prob}, mean, st\_dev)$