Capacity Allocation

Outline

- Two-class allocation
- Multiple-class allocation
- Demand dependence
- Bid prices

Based on Phillips (2005) Chapter 7
Booking Limits for 2 Fare Classes

Only one decision variable: \( b = b_2 \)
- Second class has booking limit \( b \)
- The first class has protection level of \( C-b \)

Demands for
- Full-fare class \( d_f \) whose cumulative density is \( F_f \);
- Discount fare class \( d_d \) whose cumulative density is \( F_d \).

Prices for
- Full-fare class \( p_f \);
- Discount fare class \( p_d \).
2-class Problem
Increasing the Booking Limit

$F_d(b)$: No second class is rejected with $b$ or $b+1$
Remaining capacity for the first class is the same
$d_d \leq b$ discount demand too low to be limited

1-$F_d(b)$: Remaining capacity for the first class is less by 1
$d_d > b$

1-$F_f(C-b)$: First class rejected because of $b+1$
$d_f > C-b$

$F_f(C-b)$: Low first class demand
$d_f \leq C-b$

Marginal effect on revenue

$p_d - p_f < 0$

$p_d - 0 > 0$

0

Expected marginal effect on revenue

$$= F_d(b)0 + [1 - F_d(b)] [1 - F_f(C - b)](p_d - p_f) + F_f(C - b)p_d$$

$$= [1 - F_d(b)] [p_d - [1 - F_f(C - b)]p_f]$$

$$= [1 - F_d(b)] p_f [F_f(C - b) - (1 - p_d / p_f)]$$
2 Fare Classes
Booking Limit and Protection Level

The critical term is $F_f(C-b)-(1-p_d/p_f)$

$F_f(C-b)$ decreases as $b$ increases because $F_f$ is cumulative density and increases in its argument.

If $b=0$ and still $F_f(C-b)\leq (1-p_d/p_f)$, set $b=0$.

In this case, the discount price is very low so it is not worth opening up the discount class.

For any $b$, increase $b$ as long as $F_f(C-b)>(1-p_d/p_f)$.

For a continuous full fare demand distribution,

$C-b^*=F_f^{-1}(1-p_d/p_f)$ and $b^* \leq C$.

\[
\begin{align*}
    b^* &= \max \left[ C - F_f^{-1} \left( 1 - p_d / p_f \right), 0 \right] \\
    y^* &= C - \max \left[ C - F_f^{-1} \left( 1 - p_d / p_f \right), 0 \right] \\
    &= C + \min \left[ - C + F_f^{-1} \left( 1 - p_d / p_f \right), 0 \right] \\
    &= \min \left[ F_f^{-1} \left( 1 - p_d / p_f \right), C \right] \\
    \text{or } p_d / p_f &= P(d_f > y^*).
\end{align*}
\]
Suppose that the first class (full-fare) demand is uniform between 20 and 40 seats on a flight. Moreover, first class passengers pay $1000, while the discount fare is $300.

What are booking limits for discount fare if the plane has C=100 or C=30 seats?

Protection level y first:

\[ F_f(x) = \frac{x-20}{20} \text{ for } 20 < x < 40. \]
\[ F_f^{-1}(u) = 20 + 20u \text{ for } 0 < u < 1. \]
\[ F_f^{-1}(1 - \frac{p_d}{p_f}) = F_f^{-1}(\frac{7}{10}) = 34. \]

If the capacity is 100, the booking limit is 100-34=66 for the discount class.

If the capacity is 30, the booking limit is (30-34)^+ = 0 for the discount class.
2 Fare Classes
Protection Level with Normal Demand

Suppose that the first class (full-fare) class demand is normal with mean 30 seats and standard deviation 5. Moreover, first class passengers pay $1000, while the discount fare is $300. What are booking limits for discount fare if the plane has C=100 or C=30 seats?

Standard cumulative normal distribution $\Phi(z)=\text{probability.}$

$\Phi_f^{-1}(7/10)=\text{norminv}(0.7, \text{mean}=0, \text{stdev}=1)=0.52=z.$

Protection level mean+$z*\text{stdev}=30+(0.52)5=32.6.$

Or use $\text{norminv}(0.7, \text{mean}=30, \text{stdev}=5)=32.6.$

Let us round up the protection level to 33.

If the capacity is 100, the booking limit is 100-33=67 for the discount class.

If the capacity is 30, the booking limit is (30-33)⁺=0 for the discount class.
Multiple Fare Classes
Nested Protection Levels

- Fare classes indexed by \( j = 1, 2, \ldots, n \). \( j = 1 \) is the highest class, while \( j = n \) is the lowest class.
- Protection level \( y_i \) for class \( i \) protects future reservations or classes \( j = \{1, 2, \ldots, i\} \).

<table>
<thead>
<tr>
<th>Protection levels</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1,..,n}</td>
</tr>
<tr>
<td>{1,..,n-1}</td>
</tr>
<tr>
<td>{1,2,3}</td>
</tr>
<tr>
<td>{1,2}</td>
</tr>
<tr>
<td>{1}</td>
</tr>
</tbody>
</table>

- Periods
- \( n \) \( n-1 \) 3 2 1
- Independent Demands
- \( d_n \) \( d_{n-1} \) \( d_3 \) \( d_2 \) \( d_1 \)
- Monotone Prices
- \( p_n < p_{n-1} \) \( p_3 < p_2 < p_1 \)
- Time-wise Separated Bookings
- \( x_n \) \( x_{n-1} \) \( x_3 \) \( x_2 \) \( x_1 \)

Check-in Period=0
Multiple Fare Classes
Increasing the Booking Limit

$F_3(b_3)$: No third class is rejected with $b_3$ or $b_3+1$
Remaining capacity for first/second class is the same

- $d_3 \leq b_3$
  - $d_3 > b_3$

-$F_3(b_3)$: Remaining capacity for first/second class is less by 1

$1-F_3(b_3)$: Remaining capacity for first/second class is the same

- First class rejected because of $b_3+1$
  - $p_3-p_1 < 0$
  - $p_3-p_2 < 0$
  - $p_3-0 > 0$
  - 0

Challenge: Computing the probability that first or second class is rejected by increasing the booking limit of the third class.
Expected Marginal Seat Revenue (EMSR-a)
Version-a Heuristic: Protection Level Decomposition

- In period 3, when accepting bookings for class 3, protect class 1 and class 2.
- Let $y_{1,3}$ and $y_{2,3}$ be the protection levels of first and second classes against the third class.
- We can set
  \[ y_{1,3} = F_1^{-1}(1 - p_3 / p_1) \quad \text{and} \quad y_{2,3} = F_2^{-1}(1 - p_3 / p_2) \]
- The protection level for both first and second classes while booking for class 3 is
  \[ y_2 = y_{1,3} + y_{2,3} \]

- In general,
  \[ y_i = y_{1,i+1} + y_{2,i+1} + \ldots + y_{i,i+1} = \sum_{j=1}^{i} y_{j,i+1} \]
  where $y_{j,i+1}$ is the protection level of class $j$ from class $i+1$. That is,
  \[ y_{j,i+1} = F_j^{-1}(1 - p_{i+1} / p_j) \quad \text{or} \quad p_{i+1} / p_j = P(d_j > y_{j,i+1}) \]
- Protect class $j$ more against class $i+1$ if
  - Class $j$ demand is larger
  - Class $j$ price is larger
Expected Marginal Seat Revenue (EMSR-b)
Version-b Heuristic: Remaining Demand Aggregation

- In period 3, when accepting bookings for class 3, protect class 1 and class 2.
- Aggregate classes 1 and 2 into a new demand class \{1,2\}.
- Set demand and price for the class \{1,2\}:
  \[ d_{\{1,2\}} = d_1 + d_2 \quad \text{and} \quad p_{\{1,2\}} = \frac{p_1 E_1 + p_2 E_2}{E_1 + E_2} \]
- The protection level for the aggregate class \{1,2\} while booking for class 3 is
  \[ p_3 / p_{\{1,2\}} = P(d_{\{1,2\}} > y_2) \]

- In general,
  \[ d_{\{1,2,..,i\}} = d_1 + d_2 + ... + d_i \quad \text{and} \quad p_{\{1,2,..,i\}} = \sum_{j=1}^{i} p_j \frac{E_{d_j}}{\sum_{k=1}^{i} E_{d_k}} \]
  \[ p_{i+1} / p_{\{1,2,..,i\}} = P(d_{\{1,2,..,i\}} > y_i) \]
- Protect class \{1,2,..,i\} more against class i+1 if
  - Class \{1,2,..,i\} demand is larger
  - Class \{1,2,..,i\} price is larger
Version-b Heuristic with Normal Demands

- With normal independent demands mean $\mu_i$ and standard deviation $\delta_i$

$$
\mu_{\{1,2,\ldots,i\}} = \mu_1 + \mu_2 + \ldots + \mu_i, \quad \delta_{\{1,2,\ldots,i\}} = \sqrt{\sum_{j=1}^{i} \delta_j^2} \quad \text{and} \quad p_{\{1,2,\ldots,i\}} = \frac{\sum_{j=1}^{i} p_i d_i}{\mu_{\{1,2,\ldots,i\}}}
$$

- The protection level for the aggregate class $\{1,2,\ldots,i\}$ while booking for class $i+1$ is

$$
y_{\{1,2,\ldots,i\}} = \text{Norminv}(1-p_{i+1}/p_{\{1,2,\ldots,i\}}, \mu_{\{1,2,\ldots,i\}}, \delta_{\{1,2,\ldots,i\}})
$$

- Example: Suppose that first and second class demands are normal with the same parameters $\mu_1 = \mu_2 = 20; \quad \delta_1 = \delta_2 = 5$

What is the distribution of the demand of aggregate class $\{1,2\}$?

The aggregate distribution is also Normal with parameters:

$$
\mu_{\{1,2\}} = \mu_{\{1,2\}} = 40; \quad \delta_{\{1,2\}} = \sqrt{5^2 + 5^2} \approx 7.1
$$
Comparison of Version-a and Version-b Heuristics under Normal Demands

- Example: Suppose that first and second class demands are normal with the same parameters \( \mu_1 = \mu_2 = 20; \ \delta_1 = \delta_2 = 5. \)

Then the distribution of the demand of aggregate class \( \{1,2\} \) is also Normal with parameters:

\[
\mu_{\{1,2\}} = 40; \ \delta_{\{1,2\}} = \sqrt{5^2 + 5^2} \approx 7.1
\]

Furthermore, suppose that the prices for classes are \( p_1=1000; \ p_2=700; \ p_3=400. \) Find the protection level for classes 1 and 2 by using EMSR-a and EMSR-b heuristics.

**EMSR-a heuristic:**
- The protection level for class 1 from class 3:
  \[ y_{1,3} = \text{norminv}(1-400/1000, 20, 5) = 21.27 \]
- The protection level for class 2 from class 3:
  \[ y_{2,3} = \text{norminv}(1-400/700, 20, 5) = 19.10 \]
- The protection level for classes 1 and 2 from class 3:
  \[ y_2 = 21.27 + 19.10 = 40.37 \]

**EMSR-b heuristic:**
- Price for the aggregate class \( \{1,2\} = 850. \)
- The protection level for the aggregate class \( \{1,2\} \) while booking for class 3 is
  \[ y_{\{1,2\}} = \text{norminv}(1-400/850, 40, 7.1) = 40.52 \]
- The protection level for classes 1 and 2 from class 3:
  \[ y_2 = 40.52 \]

The protection levels above are almost the same.
2-class Problem
Demand Dependence By Buying up

Marginal effect on revenue

$F_d(b)$: No second class is rejected with $b$ or $b+1$

$1-F_d(b)$: Remaining capacity for first class is the same

$d_d \leq b$

$p_d - p_f < 0$

$d_d > b$

$1-F_f(C-b)$: First class rejected because of $b+1$

$d_f > C-b$

$p_d - p_f < 0$

$d_f \leq C-b$

$F_f(C-b)$: Low first class demand

$p_d - 0 > 0$

$1-\alpha$

$\alpha$: Rejected class 2 buys up class 1

$p_d - p_f < 0$

Increasing $b$, we lose the opportunity of selling at $p_f$ to low-paying customers who buy up when their class is closed

Expected marginal effect on revenue

$$= F_d(b)0 + [1 - F_d(b)] \left[ [1 - F_f(C-b)](p_d - p_f) + F_f(C-b)(p_d - \alpha p_f) \right]$$

$$= [1 - F_d(b)] \left[ p_d - \alpha p_f - [1 - F_f(C-b)](1-\alpha)p_f \right]$$

$$= [1 - F_d(b)](1-\alpha)p_f \left[ p_d - \alpha p_f /[(1-\alpha)p_f] - [1 - F_f(C-b)] \right]$$
2 Fare Classes

Booking Limit and Protection Level

The critical term is \( F_f(C-b) + [p_d - \alpha p_f]/[(1- \alpha) p_f] - 1 \)

Or equivalently \( F_f(C-b) - [p_f p_d]/[(1- \alpha) p_f] \)

\( F_f(C-b) \) decreases as \( b \) increases.

If \( b=0 \) and still \( F_f(C-b) \leq [p_f p_d]/[(1- \alpha) p_f] \), set \( b=0 \).

In this case, either the discount price is very low or the buy up probability \( \alpha \) is high so it is not worth opening up the discount class.

For any \( b \), increase \( b \) as long as \( F_f(C-b) > [p_f p_d]/[(1- \alpha) p_f] \).

For a continuous full fare demand distribution,

\[ C - b^* = F_f^{-1}([p_f p_d]/[(1- \alpha) p_f]) \quad \text{and} \quad b^* \leq C. \]

For optimal protection level for the first class

\[ C - F_f^{-1}([p_f p_d]/[(1- \alpha) p_f]) \]

\[ F_f^{-1}([p_f p_d]/[(1- \alpha) p_f]) \]

For optimal booking limit for the second class

\[ C - F_f^{-1}([p_f p_d]/[(1- \alpha) p_f]) \]

\[ [p_f p_d]/[(1- \alpha) p_f] \]

\[ F_f^{-1}([p_f p_d]/[(1- \alpha) p_f]) \]

\[ p_d - \alpha p_f = p_d - \alpha p_f = P(d_f > y^*). \]

\[ b^* = \max \left[ C - F_f^{-1}([p_f - p_d]/[(1- \alpha) p_f]), 0 \right] \]

\[ y^* = \min \left[ F_f^{-1}([p_f - p_d]/[(1- \alpha) p_f]), C \right] \]

\[ \min \left[ F_f^{-1}([p_f - p_d]/[(1- \alpha) p_f]), C \right] \]

\[ \max \left[ C - F_f^{-1}([p_f - p_d]/[(1- \alpha) p_f]), 0 \right] \]
2 Fare Classes
Protection Level with Uniform Demand

Suppose that the first class (full-fare) class demand is uniform between 20 and 40 seats on a flight. Moreover, first class passengers pay $1000, while the discount fare is $300. Buy up probability is 0.2.

What are booking limits for discount fare if the plane has C=100 or C=30 seats?

Protection level \( y \) first:
\[
F_f(x) = \frac{(x-20)}{20} \text{ for } 20 < x < 40.
\]
\[
F_f^{-1}(u) = 20 + 20u \text{ for } 0 < u < 1.
\]
\[
F_f^{-1}(\frac{p_f - p_d}{(1 - \alpha) p_f}) = 37.5.
\]

Let us round up the protection level to 38.

If the capacity is 100, the booking limit is 100-38=62 for the discount class.

If the capacity is 30, the booking limit is (30-38)^+\text{=}0 for the discount class.

Booking limit with buy-up probability 0 is 67, it decreases to 62 with buy-up probability of 0.2.
2 Fare Classes
Protection Level with Normal Demand

Suppose that the first class (full-fare) class demand is normal with mean 30 seats and standard deviation 5. Moreover, first class passengers pay $1000, while the discount fare is $300. Buy up probability is 0.2.

What are booking limits for discount fare if the plane has C=100 or C=30 seats?

Standard cumulative normal distribution $\Phi(z) = \text{ probability.}$

$\Phi_f^{-1}(7/8) = \text{ norminv}(7/8, \text{ mean}=0, \text{ stdev}=1) = 1.15 = z.$

Protection level mean $+ z \times \text{ stdev} = 30 + (1.15)5 = 35.75.$

Or use $\text{ norminv}(7/8, \text{ mean}=30, \text{ stdev}=5) = 35.75.$

Let us round up the protection level to 36.

If the capacity is 100, the booking limit is $100 - 36 = 64$ for the discount class.

If the capacity is 30, the booking limit is $(30 - 36)^+ = 0$ for the discount class.
Bid (Displacement) Prices

- A bid price for a flight is the threshold price that depends on the remaining capacity and the time until departure, such that a booking request is
  - accepted if its fare is more
  - rejected if its fare is less

than the threshold price. The bid price is about the marginal (opportunity) cost of capacity. The latter is known as displacement cost. Bid price can be equal to displacement cost.

- If the booking is accepted, the remaining flight capacity drops. The marginal cost of accepting a booking is about the marginal cost of the capacity $b$, which in a static setting would be

  \[ \frac{d}{db} \Pi(b) \approx \Pi(b) - \Pi(b - 1) \]

- In the dynamic setting, the marginal cost of capacity when $t$ units of time left is

  \[ \frac{d}{db} \Pi(b, t) \approx \Pi(b, t) - \Pi(b - 1, t) \]

  Displacement cost

- Marginal cost of capacity and bid price decrease with large capacity (supply) and with less amount of remaining time (demand)

- Opportunity cost of capacity depends on the capacity $b$.
  - If $b$ is very large, the opportunity cost is low: Not enough customers to sell the large capacity after denying the current.

- Opportunity cost of capacity also depends on the time $t$ that remains to sell the capacity.
  - If $t$ is very small, the opportunity cost is low: Not enough time to sell after denying the current.
Bid price is not actually a price, rather it is a cost. Add profit margin on the bid price to find the actual price to quote to the customer. How much profit margin?

In comparison to fare classes, booking by bid prices

- Is simpler as there is a single bid price for a single flight while there are many fare classes
- Is more complex as the bid prices must be computed dynamically while fare class booking limits can be static.

*No free lunch* by booking via fare classes or bid prices.
Towards Bid Price Computation

◆ Ex: Suppose that \( d(p) = 200 - 10p \) if \( 0 \leq p \leq 20 \). Find total margin maximizing price when the cost is \( c = 0 \) and the capacity constraint is \( b = 20 \).
  - \( b = 20 \)
    - From max \( pd(p) = 200p - 10p^2 \), the unconstrained price \( p_0 = 10 \). But \( d(10) = 100 > 20 = b \).
    - To sell all of the capacity \( 20 = b = d(p) = 200 - 10p \) or \( p = 18 \). \( \Pi(b = 20) = 360 \).
  - After selling 1, \( b = 19 \)
    - From max \( pd(p) = 200p - 10p^2 \), the unconstrained price \( p_0 = 10 \). But \( d(10) = 100 > 19 = b \).
    - To sell all of the capacity \( 19 = b = d(p) = 200 - 10p \) or \( p = 18.1 \). \( \Pi(b = 19) = 343.9 \).
    - \( \Pi(20) - \Pi(19) = 360 - 343.9 = 16.1 \). This price is different from 18 because of discreteness, see the next page. Suppose we set \( p = 18 \) slightly higher than 16.1.

◆ When capacity is binding, bid price \( \approx \) optimal price, subject to discreteness of sellable capacity units
  - When capacity is not discrete, but continuous at small \( \epsilon \), bid price is \( \frac{\Pi(b) - \Pi(b - \epsilon)}{\epsilon} \), which is \( \frac{\partial \Pi(b)}{\partial b} \).
    » When the capacity is binding, \( \Pi(b) = bp \) and \( p = d^{-1}(p) \) so \( \frac{\partial \Pi(b)}{\partial b} = \frac{\partial}{\partial b} bp = p = d^{-1}(p) \).
  - In the discrete example of the previous page \( \frac{\Pi(b) - \Pi(b-1)}{1} = 16.1 \) while \( d^{-1}(p) = 18 \).
Bid Prices or Optimal Prices?

◆ When capacity is non-binding, bid price = 0 < optimal price
  - Value-based pricing: Optimal price
  - (Opportunity) Cost-based pricing: Bid price
    » Opportunity cost of one fewer unit of capacity is nothing if capacity > demand

◆ Assume that future prices are equal to the current price and solve: Exemplified on the last and next pages
◆ In reality, future prices can respond to current demand, so they should be different from the current one
  - Future prices are anticipatory with respect to current demand
◆ To allow for future prices to respond, we need a recursive revenue computation.
  - \( \Pi(b, t) \) is optimal profit obtained by selling \( b \) capacity to \( h \) and \( l \) type customers.
  - \( h \) and \( l \) customers are high-type (full fare) and low-type (discount fare).
  - \( WTP_h \) and \( WTP_l \) are willingness to pay of \( h \) and \( l \) customers
  - \( q_h \) and \( q_l \) are probability of \( h \) and \( l \) customers showing up, \( q_h + q_l \leq 1 \)
  - \( p_h \) and \( p_l \) are current prices for \( h \) and \( l \) customers

\[
\Pi(b, t) = \max_{p_h, p_l} q_h P(WTP_h > p_h) p_h + q_l P(WTP_l > p_l) p_l + (q_h P(WTP_h > p_h) + q_l P(WTP_l > p_l)) \Pi(b - 1, t - 1) + (1 - q_h P(WTP_h > p_h) - q_l P(WTP_l > p_l)) \Pi(b, t - 1)
\]

Current revenue from selling to \( h \) or \( l \) customer

\( \Pi(0, t) = \Pi(b, 0) = 0 \)

- Above formulation is an example of obtaining profits over \( t \) periods but beyond the scope of this course.
Prices over Time

- With \( t = 20 \) days until flight departure/room check-in/car pick-up, decide on the price:
  - \( d(p, t = 20) = 200 - 10p \) and \( b(t = 20) = 20 \) yields \( p(t) = 18 \).
- Demand for the remaining \( t \) days can be given as \( d(p, t) = 10t - 10p \).

- Suppose 1 person buys on day 20, so \( b(t = 19) = 19 \).
- On day 19, \( b(19) = 19 \) and \( d(p, 19) = 190 - 10p \).
  - For \( b(19) = 19 \),
    - \( \max_p pd(p, 19) = 190p - 10p^2 \Rightarrow p_0 = 9.5 \). But \( d(9.5, 19) = 95 > 19 \).
    - To sell all of the capacity \( 19 = 190 - 10p \) or \( p = 17.1 \). \( \Pi(b = 19, t = 19) = 324.9 \).
  - For \( b(19) = 18 \),
    - From \( \max_p pd(p) = 190p - 10p^2 \Rightarrow p_0 = 9.5 \). But \( d(9.5, 19) = 95 > 18 \).
    - To sell all of the capacity \( 18 = 190 - 10p \) or \( p = 17.2 \). \( \Pi(b = 18, t = 19) = 309.6 \).
    - \( \Pi(19, 19) - \Pi(18, 19) = 324.9 - 309.6 = 15.3 \). Suppose we set \( p = 17 \) slightly higher than 15.3.
  - Suppose nobody buys at \( p = 17 \).

- On day 18, \( b(18) = 19 \) and \( d(p, 18) = 180 - 10p \).
  - For \( b(18) = 19 \),
    - From \( \max_p pd(p) = 180p - 10p^2 \Rightarrow p_0 = 9 \). But \( d(9, 18) = 90 > 19 = b(18) \).
    - To sell all of the capacity \( 19 = 180 - 10p \) or \( p = 16.1 \). \( \Pi(b = 19, t = 18) = 305.9 \).
  - For \( b(18) = 18 \),
    - From \( \max_p pd(p) = 180p - 10p^2 \Rightarrow p_0 = 9 \). But \( d(9, 18) = 90 > 18 = b(18) \).
    - To sell all of the capacity \( 18 = 180 - 10p \) or \( p = 16.2 \). \( \Pi(b = 18, t = 18) = 291.6 \).
    - \( \Pi(18, 18) - \Pi(17, 18) = 324.9 - 309.6 = 14.3 \). Suppose we set \( p = 16 \) slightly higher than 14.3.
  - Three buy at 16 on day 18, and so on …
Summary

- Two-class allocation
- Multiple-class allocation
- Demand dependence
- Bid prices
Normal Density Function

Excel statistical functions:
Density function (pdf) at $x$ : \texttt{normdist}(\textit{x}, \textit{mean}, \textit{st _ dev},0)
Cumulative function (cdf) at $x$ : \texttt{normdist}(\textit{x}, \textit{mean}, \textit{st _ dev},1)
Cumulative Normal Density

Excel statistical functions:
Cumulative function (cdf) at x: normdist(x, mean, st_dev, 1)
Inverse function of cdf at "prob": norminv(prob, mean, st_dev)