Surface Integrals

Surface Area \( = \int \int dA = \int \int \sec \gamma \, dx \, dy \)

\[
\sec \gamma = \frac{\sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2}}{|\partial \phi / \partial z|}
\]
5-5.2

Find the surface area cut from the cone $2x^2 + 2y^2 = 5z^2, \ z > 0$, by the cylinder $x^2 + y^2 = 2y$.

\[
\phi = -2x^2 - 2y^2 + 5z^2 \Rightarrow \phi_x = -4x, \ \phi_y = -4y, \ \phi_z = 10z
\]

therefore,

\[
\sec \gamma = \frac{\sqrt{16x^2 + 16y^2 + 100z^2}}{10z} = \frac{\sqrt{8(2x^2 + 2y^2) + 100z^2}}{10z} = \frac{\sqrt{140z^2}}{10z} = \frac{\sqrt{35}}{5}
\]

But $x^2 + y^2 = 2y \Rightarrow x^2 + (y - 1)^2 = 1$ is a circle of radius 1 centered at (0,1). Thus,

\[
S = \int \int \frac{\sqrt{35}}{5} dA = \frac{\sqrt{35}}{5} \cdot \pi \cdot 1 = \pi \sqrt{7/5}
\]
5-5.4

Find the area of the part of the cone $x^2 + y^2 = 2z^2$ in the first octant, cut out by the planes $y = 0$, and $y = x/\sqrt{3}$, and the cylinder $x^2 + y^2 = 4$.

\[ \phi = 2z^2 - x^2 - y^2 \Rightarrow \phi_x = -2x, \ \phi_y = -2y, \ \phi_z = 4z \]

therefore,

\[
\sec \gamma = \frac{\sqrt{4x^2 + 4y^2 + 16z^2}}{4|z|} = \frac{\sqrt{4(x^2 + y^2) + 16z^2}}{4z} = \frac{\sqrt{24z^2}}{4z} = \sqrt{3/2}
\]

But the triangle formed by $y = 0$ and $y = x/\sqrt{3}$ as two of its sides has included angle with tangent $\frac{1}{\sqrt{3}}$ and so value $30^\circ = \pi/6 = 2\pi/12$. Thus the area covered is $1/12$-th the area of a circle. Hence:

\[ A = \int \int \sqrt{3/2} \ dA = \sqrt{3/2} \cdot (1/12) \cdot \pi \cdot 2^2 = \pi/\sqrt{6} \]
Find the area of the part of the cone $z^2 = 3(x^2 + y^2)$ which is in the sphere $x^2 + y^2 + z^2 = 16$.

\[
\phi = z^2 - 3x^2 - 3y^2 \Rightarrow \phi_x = -6x, \quad \phi_y = -6y, \quad \phi_z = 2z
\]

therefore,

\[
\sec \gamma = \frac{\sqrt{36x^2 + 36y^2 + 4z^2}}{2|z|} = \frac{\sqrt{48(x^2 + y^2)}}{2\sqrt{3(x^2 + y^2)}} = 2
\]

Now we look at the picture and see we have to find out where the cone and sphere meet. Substituting $z^2$ from the cone into the sphere equation we have $x^2 + y^2 + 3(x^2 + y^2) = 16 \Rightarrow x^2 + y^2 = 4 = 2^2$ which is within the sphere of radius 4, so

\[
A = \int \int 2 \, dx \, dy = 2 \int_0^{2\pi} \int_0^2 r \, dr \, d\theta = 2 \cdot 2\pi \cdot \left[ r^2 / 2 \right]_0^2 = 8\pi
\]

on each half of the cone.