**4-8 SOME SPECIAL AVERAGES OF FUNCTIONS OF TWO RANDOM VARIABLES**

**Joint Moments**

The $m$th joint moments of two random variables $X$ and $Y$ are defined as

$$m_{mn} = E[X^m Y^n], m, n = 1, 2, \ldots$$  \hspace{1cm} (4-37)

Special cases of (4-37) include the means of $X$ and $Y$ obtained, respectively, by setting $m = 1$ and $n = 0$ and $m = 0$ and $n = 1$. Note that the joint moments of statistically independent random variables factor for all $m$ and $n$.

**Example 4-13**

Find the joint moments of the random variables with joint probability mass function shown in Figure 4-4 if

$$x_1 = 1, x_2 = 2, x_3 = 3, y_1 = 3, y_2 = 3, y_3 = 4$$

**Solution** Substituting into (4-37) and using (4-34b) for the expectation, we obtain

$$E(X^3 Y) = 1^3 3^4 \times 0.2 + 2^3 3^4 \times 0.6 + 3^3 4^4 \times 0.2$$

Several special cases are given in Table 4-3.

**Joint Central Moments**

These are obtained by first subtracting from $X$ and $Y$ their respective means and then finding the joint moments of these new random variables:

$$
\mu_{mn} = E[(X - \mu_X)^m (Y - \mu_Y)^n], m, n = 1, 2, \ldots
$$  \hspace{1cm} (4-38)

where $\mu_X$ and $\mu_Y$ are the means of $X$ and $Y$, respectively. The special cases $m = 2, n = 0$ and $m = 0, n = 2$ give the variances of $X$ and $Y$, respectively.

**Covariance**

This is a special case of the joint central moments with $m = n = 1$:

$$
C_{XY} = E[(X - \mu_X)(Y - \mu_Y)]
$$  \hspace{1cm} (4-39a)

By expanding the expectation, this can be put into the form

$$
C_{XY} = E[XY] - \mu_X \mu_Y = R_{XY} - \mu_X \mu_Y
$$  \hspace{1cm} (4-39b)

where $R_{XY}$ is called the *correlation*; it is a special case of (4-37) for $m = n = 1$.

**Correlation Coefficient**

The *correlation coefficient* is defined as

$$
\rho_{XY} = \frac{C_{XY}}{\sigma_X \sigma_Y}
$$  \hspace{1cm} (4-40)

where $C_{XY}$ denotes the covariance (4-39a).
Example 4-3

Consider the function of two variables
\[ f_{xy}(x, y) = \begin{cases} Axy, & 0 < x < y, \quad 0 < y < 1 \\ 0, & \text{otherwise} \end{cases} \]

(a) Find \( A \) such that this is a proper pdf.
(b) Find the probability that \( 0 < X < 0.5 \) and \( 0.5 < Y < 1 \).
(c) Obtain the marginal pdf's for \( X \) and \( Y \).

Solution  
(a) Since the volume under the joint pdf must be 1 [see the remark after (4-10)], we compute
\[ \int_0^1 \int_0^y Axy \, dx \, dy = \int_0^1 \left[ \frac{A}{2} y^2 \right]_0^y \, dy = \frac{A}{2} \int_0^1 y^2 \, dy = \frac{A}{8} = 1 \]
from which we deduce that \( A = 8 \). In the integration of the \( f_{xy}(x, y) \), we have made use of the fact that the function is nonzero over a triangle defined by \( x \) between the \( y \)-axis and the line \( x = y \) and on \( y \) from 0 to 1. The student should sketch this area and deduce that the limits of integration are proper.

(b) From (4-13), the desired probability may be computed as
\[ P(0 < X < 0.5, 0.5 < Y < 1) = 8 \int_0^{0.5} \int_0^y xy \, dx \, dy = 0.375 \]

(c) The marginal pdf for \( X \) is obtained by integrating the joint pdf over all \( y \) as given by (4-12a):
\[ f_x(x) = \int_x^1 8xy \, dy = 8x \frac{y^2}{2} \bigg|_x^1 = 4x(1 - x^2), \quad 0 < x < 1 \]
and zero elsewhere. The limits of integration are deduced by noting the region in the \( x-y \) plane, where \( f_{xy}(x, y) \) is nonzero. The marginal pdf for \( Y \) is obtained by integrating the joint pdf over all \( x \) as given by (4-12b):
\[ f_y(y) = \int_0^y 8xy \, dx = 8y \frac{x^2}{2} \bigg|_0^y = 4y^3, \quad 0 < y < 1 \]
and zero elsewhere. Note that both \( f_x(x) \) and \( f_y(y) \) integrate to 1, as they should.

Example 4-14

Find the correlation, covariance, and correlation coefficient for the random variables with joint pdf given in Example 4-3.

Solution  
By definition, the correlation is
\[ R_{xy} = E[XY] = \int_0^1 \int_0^y xy(8xy) \, dx \, dy = \frac{4}{9} \]
The means are
\[ \mu_x = \int_0^1 x[4x(1 - x^2)] \, dx = \frac{8}{15} \quad \text{and} \quad \mu_y = \int_0^1 y(4y^3) \, dy = \frac{4}{5} \]
where the marginal pdf's obtained in Example 4-3 were used. The covariance is
\[ C_{xy} = \frac{4}{9} - \frac{8}{15} \cdot \frac{4}{5} = \frac{4}{225} \]
To get the correlation coefficient, we need the variances. We compute the mean-square values first. They are
\[ E[X^2] = \int_0^1 x^2[4x(1 - x^2)] \, dx = \frac{1}{3} \quad \text{and} \quad E[Y^2] = \int_0^1 y^2(4y^3) \, dy = \frac{2}{3} \]
Thus the variances are
\[ \sigma_x^2 = \frac{11}{225} \quad \text{and} \quad \sigma_y^2 = \frac{2}{75} \]
Substituting into (4-40), we find the correlation coefficient to be
\[ \rho_{xy} = \frac{4/225}{(\sqrt{11/15})(\sqrt{2/5})} = \frac{2\sqrt{2/33}}{\sqrt{2/3}} = 0.4924 \]