Landmarks in Graphs

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Abstract

Navigation can be studied in a graph-structured framework in which the navigating agent (which we shall assume to be a point robot) moves from node to node of a “graph space”. The robot can locate itself by the presence of distinctively labeled “landmark” nodes in the graph space. For a robot navigating in Euclidean space, visual detection of a distinctive landmark provides information about the direction to the landmark, and allows the robot to determine its position by triangulation. On a graph, however, there is neither the concept of direction nor that of visibility. Instead, we shall assume that a robot navigating on a graph can sense the distances to a set of landmarks.

Evidently, if the robot knows its distances to a sufficiently large set of landmarks, its position on the graph is uniquely determined. This suggests the following problem: given a graph, what are the fewest number of landmarks needed, and where should they be located, so that the distances to the landmarks uniquely determine the robot’s position on the graph? This is actually a classical problem about metric spaces. A minimum set of landmarks which uniquely determine the robot’s position is called a “metric basis”, and the minimum number of landmarks is called the “metric dimension” of the graph. In this paper we present some results about this problem. Our main new results are that the metric dimension of a graph with \( n \) nodes can be approximated in polynomial time within a factor of \( O(\log n) \), and some properties of graphs with metric dimension two.

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1 Introduction

Consider a robot which is navigating in a space modeled by a graph, and which wants to know its current location. It can send a signal to find out how far it is from each among a set of fixed landmarks. We study the problem of computing the minimum number of landmarks required, and where they should be placed, such that the robot can always determine its location. The set of nodes where the landmarks are placed is called the metric basis of the graph, and the number of landmarks is called the metric dimension of the graph.

We now give a formal definition of the notion of metric dimension. The metric dimension of a graph \( G = (V, E) \) is the cardinality of a smallest subset \( S \subseteq V \), such that, for each pair of vertices \( u, v \in V \), there is a \( w \in S \) such that the length of a shortest path from \( w \) to \( u \) is different from the length of a shortest path from \( w \) to \( v \).

We associate “coordinates” with each node based on the distances from the node to the landmarks. Our goal is to pick just enough landmarks so that each node has a unique tuple of coordinates. For example, in Euclidean \( d \)-space, it is easy to show that any set of \( d + 1 \) points in general position constitutes a metric basis.

Let \( G = (V, E) \) be a connected, undirected graph. A “coordinate system” on \( G \) is defined as follows. We pick a set of nodes as the metric basis; each node in the basis corresponds to a landmark. For each landmark, the coordinate of each node \( v \in V \) in the corresponding “dimension” is equal to the length of a shortest path from the landmark to \( v \). Thus for a metric basis, each node has a vector of coordinates, a tuple of non-negative integers specifying the distances to that node from the nodes in the basis.

**Definition 1** The metric dimension of the graph \( G \) is denoted by \( \beta(G) \).

For example, a path has metric dimension 1, a cycle has metric dimension 2, and a complete graph on \( n \) nodes has metric dimension \( n - 1 \).

We first note a simple property of shortest paths on graphs.

**Proposition 1.1** Let \( G = (V, E) \) be an arbitrary graph. Let \( u, v \) and \( w \) be nodes of \( G \) and let \( \{u, v\} \in E \). Let \( d \) be the length of a shortest path from \( u \) to \( w \) in \( G \). Then the length of a shortest path from \( v \) to \( w \) is one of \( \{d - 1, d, d + 1\} \).

**Related Work:** The problem of finding the metric dimension of a graph was first studied by Harary and Melter [2]. They gave a characterization for the metric dimension of trees; their proof however has an error (more specifically, the proof of Lemma 1 has an error). We give a similar characterization for metric dimension of trees.

Melter and Tomescu [5] studied the metric dimension problem for grid-graphs induced by lattice points in the plane when the distances are measured in the \( L_1 \) and \( L_\infty \) metrics. They showed that the metric dimension of lattice points inside a rectangle whose sides are parallel to the axes in 2-dimensional space is 2 under the \( L_1 \) metric and is 3 under the \( L_\infty \) metric. In addition, they characterized all metric bases of such a grid. They also showed that the metric dimension may be arbitrarily large if the sides of the rectangle are not parallel to the axes. In this paper, we generalize one of their results and provide a characterization for metric dimension of lattice points contained inside a \( d \)-dimensional rectangle under the \( L_1 \) metric.

We then consider graphs having small metric dimension; and show that a graph has metric dimension 1 iff it is a path. Garey and Johnson (unpublished result, cited in [1]) proved that this problem was NP-complete for general graphs by a reduction from 3-dimensional matching.
completeness, we provide a reduction from 3-SAT in the appendix. By providing an approximation preserving reduction to the set cover problem, we then show that the metric dimension of a graph may be approximated in polynomial time within a factor of $O(\log n)$.

2 Metric dimension of special graphs

2.1 Trees

In this section, we study the problem of computing the metric dimension of trees. We show that this problem can be solved efficiently in linear time. Let $T = (V, E)$ be an arbitrary tree on $n$ nodes. We will assume that $T$ is not just a path; we will show later that the metric dimension of a path is 1.

Definition 2 Let $T = (V, E)$ be a tree, and $v$ a specified vertex in $T$. Partition the edges of $T$ by the equivalence relation $=_{v}$, defined as follows: two edges $e = f$ if and only if there is a path in $T$ including $e$ and $f$ that does not have $v$ as an internal vertex. The subgraphs induced by the edges of the equivalence classes of $E$ are called the bridges of $T$ relative to $v$.

Definition 3 For each node $v \in V$ of a tree $T = (V, E)$, the legs at $v$ are the bridges which are paths. We use $\ell_v$ to denote the number of legs at $v$.

![Figure 1: Example of a node with 4 legs.](image)

For example, in Fig. 1, node $u$ has 4 legs.

We now prove that the metric dimension of $T$, $\beta(T)$, is exactly

$$\sum_{v \in V : \ell_v > 1} (\ell_v - 1).$$

The characterization obtained by Harary and Melter [2] is essentially the same, with a different proof.

We first obtain a lower bound on $\beta(T)$.

Lemma 2.1 Let $T = (V, E)$ be a tree which is not a path. Then

$$\beta(T) \geq \sum_{v \in V : \ell_v > 1} (\ell_v - 1).$$
Proof. Consider any node $v$ with $\ell_v > 1$. Observe that for any metric basis all but (at most) one of $v$’s legs must contain a landmark; otherwise the neighbors of $v$ in those legs without landmarks have the same coordinates, making the configuration invalid. Therefore at least $\ell_v - 1$ landmarks must be placed on the legs of $v$. If $T$ is not a path, the legs corresponding to different nodes (each with at least two legs) are disjoint. Therefore the number of landmarks in any metric basis is at least the sum stated above.

We now obtain an upper bound on $\beta(T)$ constructively.

**Algorithm to place landmarks on a tree**

1. Compute $\ell_v$ for each node $v$.

2. Each node $v$ with $\ell_v > 1$ is allocated $(\ell_v - 1)$ landmarks. These landmarks are placed on all but one of the leaves associated with the legs of $v$.

It is easy to implement the above algorithm in linear time using a post-order traversal of the tree. Both steps of the algorithm can be completed in a single traversal of the tree. Also, the algorithm clearly uses the minimum number of landmarks necessary (as shown in Lemma 2.1). We now show that the algorithm generates a metric basis.

**Lemma 2.2** Let $T$ be rooted arbitrarily. Any node $v$ of degree greater than 2 has a descendant landmark.

Proof. In the subtree of $v$ let $w$ be a deepest node whose degree is greater than 2 ($w$ may be the same as $v$). Then $w$ has at least two legs (in the subtree of $v$) and at least one landmark is placed in the subtree of $v$.

**Lemma 2.3** The above algorithm produces a valid configuration of landmarks for a given tree $T$ (which is not a path) and uses $\sum_{q \in V}(\ell_q - 1)$ landmarks, where the sum is taken over those nodes with $\ell_q > 1$.

Proof. Root the tree $T$ at an arbitrary leaf $r$ that has a landmark. We will show that for any pair of nodes $u$ and $v$, there exists a landmark that distinguishes these two nodes. We use the notation $lca(u, v)$ to refer to the least common ancestor of vertices $u$ and $v$ in the tree $T$.

Case 1 - $u$ and $v$ are at different distances from $r$: The landmark at $r$ distinguishes $u$ from $v$.

Case 2 - $u$ and $v$ are at the same depth and at least one of $u$ or $v$ has a (not necessarily proper) descendant $w$ with degree greater than 2: By Lemma 2.2, $w$ has a descendant landmark and this landmark distinguishes $u$ from $v$.

Case 3 - $u$ and $v$ are at the same depth and neither has a descendant with degree greater than 2:

Case 3a - the path from $u$ to $v$ has only one node of degree greater than 2, namely $w = lca(u, v)$: In this case, $u$ and $v$ are on different legs of $w$. Since $w$ has at least two legs, it places landmarks on the leaves of all its legs but one. Hence at least one of these two legs receives a landmark, which distinguishes $u$ from $v$.

Case 3b - there is a node $x$ different from $w = lca(u, v)$ on the path from $u$ to $v$ and the degree of $x$ is greater than 2: The node $x$ must have a descendant landmark which distinguishes $u$ from $v$ (note that $u$ and $v$ are at the same depth and hence $w$ is equidistant from $u$ and $v$).
Theorem 2.4 Let $T = (V, E)$ be a tree which is not a path. Then

$$\beta(T) = \sum_{v \in V: \ell_v > 1} (\ell_v - 1).$$

Proof. By Lemma 2.1, this sum is a lower bound on $\beta(T)$. Lemma 2.3 shows that the same sum is also an upper bound on $\beta(T)$. \qed

2.2 Grid graphs

We now study grid graphs formed by integer lattice points in a bounded $d$-dimensional space. In 2-dimensional space, Melter and Tomescu [5] showed that the metric dimension is 2 under the $L_1$ metric for lattice points within a rectangle whose sides are parallel to the axes. Such a set of points corresponds to a 2-dimensional grid. We generalize this result to higher dimensions and show that the metric dimension of $d$-dimensional grids is $d$. Let us assume that the size of the grid is $D_1 \times D_2 \times \ldots \times D_d$.

Theorem 2.5 The metric dimension of a $d$-dimensional grid ($d \geq 2$) is $d$.

Proof. Assume we give each node a position vector which is its location in the integer lattice. We place the landmarks at the following positions. The landmark $b_0$ is kept at the origin $(0, 0, \ldots, 0)$. Let $X_i$ be the node for which the $i^{th}$ component of its position vector is $D_i$, with all other components being 0. The landmark $b_i, 1 \leq i \leq d - 1$ is kept at node $X_i$.

We will now show that each node gets a unique coordinate tuple based on its distances from this set of landmarks. Let the distance of node $v$, with position vector $(x_1, x_2, \ldots, x_d)$, from landmark $b_i$ be $d_i(0 \leq i \leq d - 1)$. We get the following equations:

$$x_1 + x_2 + \ldots + x_d = d_0$$

$$(D_1 - x_1) + x_2 + \ldots + x_d = d_1$$

$$x_1 + (D_2 - x_2) + \ldots + x_d = d_2$$

$$x_1 + x_2 \ldots + (D_{d-1} - x_{d-1}) + x_d = d_{d-1}$$

It is not difficult to see that solving these equations yields a unique solution for the position vector of node $v$. Hence each node has distinct coordinates. We leave it for the reader to see why $d$ is a lower bound on the metric dimension. \qed

3 Graphs with small metric dimension

We first investigate graphs that require only a few landmarks. We show that paths are the only graphs with $\beta = 1$. We then investigate a few properties of graphs with $\beta = 2$.

3.1 Graphs with metric dimension 1

Graphs that require only a single landmark are clearly simple in nature. We characterize them exactly.

Theorem 3.1 A graph $G = (V, E)$ has $\beta = 1$ iff $G$ is a path.
Proof. We give a proof by contradiction. Let $G$ be a graph with $\beta = 1$. Let the landmark node be vertex $u$ of $G$. First observe that the degree of $u$ is 1; otherwise the nodes adjacent to $u$ will have the same coordinate of 1. Suppose $G$ is not a path. Then it contains a node $v$ whose degree is at least 3. Let $N = \{v_1, v_2, \ldots, v_k\}$ be the neighbors of $v$. Since there is only one landmark, every node has a single coordinate (distance from the landmark). Let $d$ be the coordinate of $v$. By Proposition 1.1, the coordinates of each of the nodes in $N$ is one of $\{d - 1, d, d + 1\}$. None of the nodes in $N$ may be at a distance $d$ from the landmark since the coordinate $d$ is taken by $v$. Therefore, since $|N| \geq 3$, at least two nodes in $N$ have the same coordinate. This is a contradiction because we assumed that $\beta(G) = 1$.

We now show that if $G$ is a path then $\beta(G) = 1$. Let a landmark be placed at one of the two ends of the path. It is easily verified that this is a metric basis of the graph.

3.2 Graphs with metric dimension 2

Graphs with $\beta = 2$ have a richer structure. We study a few properties of such graphs. We show that these graphs contain neither $K_5$ nor $K_{3,3}$ as a subgraph. This might lead one to conjecture that such graphs have to be planar; but we will exhibit a non-planar graph with metric dimension 2.

Theorem 3.2 A graph $G$ with $\beta(G) = 2$ cannot have $K_5$ as a subgraph.

Proof. Consider a graph $G$ with $K_5$ as a subgraph. Let the nodes of the subgraph be $v_1, \ldots, v_5$. Suppose two landmarks are sufficient for $G$. Since every pair of nodes in $v_1, \ldots, v_5$ are adjacent in $G$, by Proposition 1.1, the first coordinate of these nodes must be one of $\{y, y + 1\}$ for some integer $y$. Similarly the second coordinate of the nodes is one of $\{z, z + 1\}$ for some $z$. With these coordinates, there are only four distinct coordinates for the five nodes, thus making the configurations of the landmarks invalid.

Remark: The proof can be extended to show that a graph $G$ with $\beta(G) = k$ cannot have $K_{2^k+1}$ as a subgraph.

Theorem 3.3 A graph $G$ with $\beta(G) = 2$ cannot have $K_{3,3}$ as a subgraph.

Proof. Assume for contradiction that $K_{3,3}$ is present as a subgraph and that there is a metric basis of size two. All nodes have been given distinct coordinates. Let the nodes of $K_{3,3}$ be $\{v_1, v_2, v_3\}$ and $\{v_4, v_5, v_6\}$ with edges going across from one set of nodes to the other. Among these six nodes, let node $v_4$ have the smallest first coordinate and have coordinates $(a, b)$. Nodes $\{v_1, v_2, v_3\}$ must all have first coordinate either $a$ or $a + 1$.

1. Suppose all three are $a + 1$. The 2nd coordinates must be $\{b - 1, b, b + 1\}$ (in some order). This forces the second coordinates of nodes $v_5$ and $v_6$ to be $b$. There is no way to assign distinct coordinates to nodes $\{v_4, v_5, v_6\}$.

2. Suppose all three are $a$. The second coordinates must be $\{b - 1, b, b + 1\}$ (in some order). There are two nodes with coordinates $(a, b)$.

3. Suppose nodes $v_1$ and $v_2$ have first coordinate $a$, and node $v_3$ has first coordinate $a + 1$. Nodes $v_1$ and $v_2$ have their second coordinates $\{b - 1, b + 1\}$ in some order. Clearly the second coordinate of nodes $v_5$ and $v_6$ is $b$. There is no way to assign distinct coordinates to nodes $\{v_4, v_5, v_6\}$.
4. Suppose node \( v_1 \) has first coordinate \( a \), and nodes \( v_2 \) and \( v_3 \) have first coordinate \( a + 1 \). Node \( v_1 \) can be either \((a, b + 1)\) or \((a, b - 1)\).

(a) Node \( v_1 = (a, b + 1) \). Nodes \( v_2 \) and \( v_3 \) have to choose their second coordinates. The choices are \( \{b, b - 1\} \) or \( \{b, b + 1\} \) or \( \{b + 1, b - 1\} \). We consider each case separately.

(i) The second coordinate of \( v_5 \) must be \( b \). There is no choice for the first.

(ii) In this case nodes \( v_5 \) and \( v_6 \) have to pick from \( \{a, a + 1\} \) for the first coordinate and \( \{b, b + 1\} \) for the second coordinate. Since there are a total of four distinct choices and nodes \( v_1, v_2 \) and \( v_3 \) have used up three of them we cannot assign coordinates to \( v_5 \) and \( v_6 \).

(iii) The second coordinate of \( v_5 \) and \( v_6 \) must be \( b \). There is no choice for the first.

(b) Node \( v_1 = (a, b - 1) \). Nodes \( v_2 \) and \( v_3 \) have to choose their second coordinates. The choices are \( \{b, b - 1\} \) or \( \{b, b + 1\} \) or \( \{b + 1, b - 1\} \). We consider each case separately.

(i) The choices for nodes \( v_5 \) and \( v_6 \) are \( \{a, a + 1\} \) for the first coordinate and \( \{b - 1, b\} \) for the second coordinate. Since there are a total of four distinct choices and nodes \( v_1, v_2 \) and \( v_3 \) have used up three of them we cannot assign coordinates to nodes \( v_5 \) and \( v_6 \).

(ii) The second coordinate of \( v_5 \) must be \( b \). There is no choice for the first.

(iii) The second coordinate of \( v_5 \) must be \( b \). The first coordinate is forced to be \( a + 1 \). There is no choice for node \( v_6 \).

\[ \square \]

**Theorem 3.4** There are non-planar graphs with metric dimension 2.

*Proof.* We give an example of a non-planar graph whose metric dimension is 2 (Fig. 2). It is easily verified that two landmarks are sufficient for this graph (place the landmarks on nodes \( u \) and \( v \)). A \( K_5 \) homeomorph of the graph is shown using bold lines, thus showing that the graph is non-planar.

The following theorem captures a few other properties of graphs with metric dimension 2.

**Theorem 3.5** Let \( G = (V, E) \) be a graph with metric dimension 2 and let \( \{a, b\} \subset V \) be a metric basis in \( G \). The following are true:

1. There is a unique shortest path \( P \) between \( a \) and \( b \).

2. The degrees of \( a \) and \( b \) are at most 3.

3. Every other node on \( P \) has degree at most 5.

*Proof.* Suppose there were two shortest paths \( P_1 \) and \( P_2 \) between \( a \) and \( b \). Consider the nodes nearest to \( a \) in which \( P_1 \) and \( P_2 \) differ, i.e., distinct nodes \( u \) and \( v \) on the two shortest paths which are both equidistant from \( a \). It is easy to verify that \( u \) and \( v \) have exactly the same coordinates, contradicting the fact that the placement of landmarks on the nodes \( a \) and \( b \) is valid. Hence the shortest path between \( a \) and \( b \) is unique.

Let the coordinates of \( a \) be \((0, x)\). All neighbors of \( a \) have 1 as their first coordinate. Therefore, applying Proposition 1.1, the coordinates of the neighbors of \( a \) must be one of \((1, x - 1), (1, x)\) or \((1, x + 1)\). Hence the degree of \( a \) is at most 3, and analogously for \( b \).
Consider any other node \( w \) on the shortest path between \( a \) and \( b \). Let its coordinates be \((p, q)\). Clearly \( p + q = x \), the distance between \( a \) and \( b \). The degree of \( w \) is at most 5 since there are no nodes with coordinates \((p - 1, q - 1)\) or \((p - 1, q)\) or \((p, q - 1)\).

The following theorem gives a lower bound on the diameter of a graph with metric dimension 2.

**Theorem 3.6** Let \( G = (V, E) \) be a graph with metric dimension \( k \) and \(|V(G)| = n\). Let \( D \) be the diameter of \( G \). Then \(|V| \leq D^k + k\).

**Proof.** Consider any valid configuration of two landmarks on \( G \). Since the diameter of \( G \) is \( D \), each coordinate of \( G \) is an integer between 0 and \( D \). Only the nodes on which landmarks were placed have one coordinate 0. Each of the remaining nodes must get a unique coordinate from one of \( D^k \) possibilities. Therefore \( G \) has at most \( D^k + k \) nodes.

### 4 Approximating the metric dimension of a graph

In this section, we show that the metric dimension of a graph with \( n \) nodes can be approximated in polynomial time within a factor of \( O(\log n) \). We show that there is an approximation preserving reduction from the problem of finding \( \beta(G) \) to the set cover problem. We can then use the \( O(\log n) \) factor approximation algorithm for the set cover problem [3, 4] to obtain an approximation algorithm for the metric dimension problem.

**Theorem 4.1** Given an arbitrary graph \( G = (V, E) \) with \( n \) nodes, then \( \beta(G) \) can be approximated within a factor of \( O(\log n) \) in polynomial time.

**Proof.** We construct an instance of the set cover problem from \( G \). The intuition is that every pair of distinct nodes must be distinguished by some landmark. We can easily compute all those
pairs of nodes that are distinguished by placing a landmark on a given node. The metric dimension problem is that of finding a set of nodes of minimum cardinality such that every pair of nodes is distinguished by some node in this set.

The elements of the universe (in the set cover problem) correspond to pairs of nodes of $G$, \( \{u,v\} : u \neq v \). For each node $v \in V$, we place the set of all pairs of nodes which are distinguished by placing a landmark at $v$ into a single subset $S_v$. Therefore there are $\binom{n}{2}$ elements and $n$ subsets in the set cover problem ($|V| = n$). It is easily verified that there is a set cover of size $k$ iff there exists a metric basis of size $k$ in $G$. Finding a set cover within a factor of $O(\log n)$ therefore yields the same approximation for the metric dimension problem.

**Acknowledgements:** We would like to thank Bob Melter for pointers into previous literature regarding metric dimensions of graphs, and for providing us with a copy of [2]. We would also like to thank David Johnson for sending us the NP-completeness proof cited in [1].

## A NP-hardness in general graphs

We now show that the problem of finding the metric dimension of an arbitrary graph is NP-hard.

**Theorem A.1** Given an arbitrary graph $G = (V, E)$ and an integer $k$, deciding whether $\beta(G) \leq k$ is NP-complete.

**Proof.** The problem is clearly in NP. We give the NP-hardness proof by a reduction from 3-SAT.

Consider an arbitrary input to 3-SAT, a formula $F$ with $n$ variables and $m$ clauses. Let the variables be $x_1, \ldots, x_n$ and the clauses be $C_1, \ldots, C_m$.

For each variable $x_i$ we construct a gadget as follows (see Fig. 3):

![Gadget for a variable](image)

Figure 3: Gadget for a variable

The nodes $T_i$ and $F_i$ are the “true” and “false” ends of the gadget. The gadget is attached to the rest of the graph only through these nodes.

Suppose $C_j = y^1_j \lor y^2_j \lor y^3_j$, where $y^k_j$ is a literal in clause $C_j$. For each such clause $C_j$ we construct a gadget as follows (see Fig. 4).

We now show the connections between the clause and variable gadgets.

If a variable $x_i$ occurs as a positive literal in clause $C_j$, we add the edges \( \{T_i, c^1_j\}, \{F_i, c^1_j\} \) and \( \{T_i, c^3_j\}. \) If it occurs in $C_j$ as a negative literal, we add the same edges, except we replace \( \{F_i, c^3_j\} \) by \( \{T_i, c^3_j\}. \) Fig. 5 shows the edges added thus corresponding to the clause $C_j = x_1 \lor \bar{x}_2 \lor x_3$. We call these the **truth testing** edges.
For all $k$ such that neither $x_k$ nor $\bar{x}_k$ occur in $C_j$, add the following edges to the generated graph: $\{T_k, c^1_j\}, \{T_k, c^2_j\}, \{F_k, c^3_j\}, \{F_k, c^4_j\}$. The reason why these edges are added is that no matter what value is assigned to $x_k$ (corresponds to placing a landmark at an appropriate location), this gives identical coordinates to both $c^2_j$ and $c^3_j$ in the gadget corresponding to clause $C_j$. We call these the neutralizing edges.

Thus the graph $G$ that is constructed from the formula $F$ with $n$ variables and $m$ clauses has $6n + 5m$ nodes. The edges of $G$ are variable gadget edges, clause gadget edges, truth testing edges and neutralizing edges. It is clear that given $F$, $G$ can be easily constructed in polynomial time.

We will now prove that $F$ is satisfiable if and only if the metric dimension of $G$ is exactly $n + m$.

We will first note a few useful properties of $G$.

**Lemma A.2** Let $x_i$ be an arbitrary variable in $F$. In any metric basis, at least one of the nodes $\{a^1_i, a^2_i, b^1_i, b^2_i\}$ must have a landmark on it.

**Proof.** Suppose none of these nodes has a landmark. Since these variables are not connected to any node other than the ones shown in the variable gadget (Fig. 3), symmetry implies that $a^1_i$ and $a^2_i$ have exactly the same coordinates. This contradicts the statement of the lemma that the placement of the landmarks is valid. □

**Lemma A.3** Let $C_j$ be an arbitrary clause in $F$. In any metric basis, at least one of the nodes $\{c^1_j, c^2_j\}$ must have a landmark on it.

**Proof.** If there is no landmark on either of these nodes, due to symmetry, these two nodes have exactly the same coordinates. This implies that the placement of landmarks is invalid. □
Corollary A.4 The metric dimension of $G$ is at least $m + n$.

Lemma A.5 If $F$ is satisfiable, the metric dimension of $G$ is $m + n$.

Proof. We know that the metric dimension is at least $m + n$. We now exhibit a metric basis of size $m + n$ based on a satisfying assignment of $F$.

Fix a satisfying assignment for $F$. For each clause $C_j$, place a landmark on $c^1_j$. For each variable $x_i$, if its value is true, place a landmark on $a^1_i$; otherwise place a landmark on $b^1_i$.

We now show that this is a metric basis. The only sets of nodes for which we need to show that they have distinct coordinates are pairs of nodes of the form $\{c^1_j, c^3_j\}$. Since the new clause gadget is a 3-clique, the only nodes of the same clause gadget are the end nodes of the clause gadget. For any other pair of nodes, it is easy to find a landmark which distinguishes between them.

For any clause $C_j$, we show that $c^1_j$ and $c^3_j$ have different coordinates if landmarks were placed based on a satisfying assignment as above. Suppose $C_j$ is satisfied by the variable $x_i$, a variable occurring as a positive literal in $C_j$ and has the value true in the assignment (the case when $x_i$ occurs as a negative literal in $C_j$ and has the value false is symmetric). Corresponding to $x_i$ being true, we placed a landmark on $a^1_i$. From this landmark, $c^1_j$ is at distance 2, while $c^3_j$ is at distance 3. Thus all nodes have distinct coordinates and therefore we have a metric basis of size $m + n$. □

Lemma A.6 If the metric dimension of $G$ is $m + n$, then $F$ is satisfiable.

Proof. Consider any metric basis of size $m + n$ in $G$. By Lemmas A.2 and A.3, we know that in any metric basis, at least one landmark must be placed within each variable and each clause gadget. Since there are exactly $m + n$ landmarks, there is exactly one landmark per variable and one landmark per clause.

We now set an assignment of the variables as follows. For each variable $x_i$, if the landmark on its gadget is on either $a^1_i$ or $a^2_i$, set $x_i$ to be true. Otherwise set $x_i$ to be false. We will now show that this yields a satisfying assignment for $F$.

Consider an arbitrary clause $C_j$. We will show that at least one of its literals is true. The main idea is in tracing which landmark distinguishes between $c^1_j$ and $c^3_j$ and showing that the corresponding variable assignment satisfies $C_j$.

For each clause $C_k$, without loss of generality, one landmark is placed on $c^1_k$. If $j = k$, both $c^1_j$ and $c^3_j$ are at distance 2 from $c^4_k$. If $j \neq k$, then due to the neutralizing edges $c^1_j$ and $c^3_j$ are at distance 4 from $c^4_k$. Therefore none of these landmarks distinguish $c^1_j$ from $c^3_j$.

For any variable $x_p$ which does not occur in $C_j$, the landmark in the variable gadget of $x_p$ is at distance 2 from each of $c^1_j$ and $c^3_j$. Therefore the only landmark that could distinguish between $c^1_j$ and $c^3_j$ must be on a variable $x_q$ which occurs in $C_j$. Due to the manner in which we have added truth testing edges, such a landmark distinguishes between the two nodes only if one of the following two statements holds.

1. $x_q$ occurs as a positive literal in $C_j$ and a landmark is placed on either $a^1_q$ or $a^2_q$; in this case $x_q$ is set to true.

2. $x_q$ occurs as a negative literal in $C_j$ and a landmark is placed on either $b^1_q$ or $b^2_q$; in this case $x_q$ is set to false.

In either case, the setting of $x_q$ is such that it satisfies $C_j$. □

Lemmas A.5 and A.6 together complete the reduction from SAT to the metric dimension problem. This completes the proof of Theorem A.1.
References


