Truth Sets and Quantifiers

We will now tie together concepts from set theory and from predicate logic. Given a predicate \( P \), and a domain \( D \), we define the truth set of \( P \) to be the set of elements \( x \) in \( D \) for which \( P(x) \) is true. The truth set of \( P(x) \) is denoted by \( \{ x \in D \mid P(x) \} \).

What are the truth sets of the predicates \( P(x) \), \( Q(x) \), and \( R(x) \), where the domain is the set of integers and \( P(x) \) is "\( |x| = 1 \)"? \( Q(x) \) is "\( x^2 = 2 \)" and \( R(x) \) is "\( |x| = x \)."

**Solution:** The truth set of \( P \), \( \{ x \in \mathbb{Z} \mid |x| = 1 \} \), is the set of integers for which \( |x| = 1 \). Because \( |x| = 1 \) when \( x = 1 \) or \( x = -1 \), and for no other integers \( x \), we see that the truth set of \( P \) is the set \( \{-1, 1\} \).

The truth set of \( Q \), \( \{ x \in \mathbb{Z} \mid x^2 = 2 \} \), is the set of integers for which \( x^2 = 2 \). This is the empty set because there are no integers \( x \) for which \( x^2 = 2 \).

The truth set of \( R \), \( \{ x \in \mathbb{Z} \mid |x| = x \} \), is the set of integers for which \( |x| = x \). Because \( |x| = x \) if and only if \( x \geq 0 \), it follows that the truth set of \( R \) is \( \mathbb{N} \), the set of nonnegative integers.

Note that \( \forall x P(x) \) is true over the domain \( U \) if and only if the truth set of \( P \) is the set \( U \). Likewise, \( \exists x P(x) \) is true over the domain \( U \) if and only if the truth set of \( P \) is nonempty.

### Exercises

1. List the members of these sets.
   a) \( \{ x \mid x \text{ is a real number such that } x^2 = 1 \} \)
   b) \( \{ x \mid x \text{ is a positive integer less than 12} \} \)
   c) \( \{ x \mid x \text{ is the square of an integer and } x < 100 \} \)
   d) \( \{ x \mid x \text{ is an integer such that } x^2 = 2 \} \)

2. Use set builder notation to give a description of each of these sets.
   a) \( \{ 0, 3, 6, 9, 12 \} \)
   b) \( \{ -3, -2, -1, 0, 1, 2, 3 \} \)
   c) \( \{ m, n, o, p \} \)

3. For each of these pairs of sets, determine whether the first is a subset of the second, the second is a subset of the first, or neither is a subset of the other.
   a) the set of airline flights from New York to New Delhi, the set of nonstop airline flights from New York to New Delhi
   b) the set of people who speak English, the set of people who speak Chinese
   c) the set of flying squirrels, the set of living creatures that can fly

4. For each of these pairs of sets, determine whether the first is a subset of the second, the second is a subset of the first, or neither is a subset of the other.
   a) the set of people who speak English, the set of people who speak English with an Australian accent
   b) the set of fruits, the set of citrus fruits
   c) the set of students studying discrete mathematics, the set of students studying data structures

5. Determine whether each of these pairs of sets are equal.
   a) \( \{ 1, 3, 3, 5, 5, 5, 5, 5, 5 \} \), \( \{ 5, 3, 1 \} \)
   b) \( \{ 1 \}, \{ 1 \} \)
   c) \( \emptyset, \{ \emptyset \} \)

6. Suppose that \( A = \{ 2, 4, 6 \} \), \( B = \{ 2, 6 \} \), \( C = \{ 4, 6 \} \), and \( D = \{ 4, 6, 8 \} \). Determine which of these sets are subsets of which other of these sets.

7. For each of the following sets, determine whether 2 is an element of that set.
   a) \( \{ x \in \mathbb{R} \mid x \text{ is an integer greater than 1} \} \)
   b) \( \{ x \in \mathbb{R} \mid x \text{ is the square of an integer} \} \)
   c) \( \{ 2, [2] \} \)
   d) \( \{ [2], [2] \} \)
   e) \( \{ [2], [2], [2] \} \)
   f) \( \{ [2], [2] \} \)

8. For each of the sets in Exercise 7, determine whether \( [2] \) is an element of that set.

9. Determine whether each of these statements is true or false.
   a) \( 0 \in \emptyset \)
   b) \( \emptyset \in \{ 0 \} \)
   c) \( \{ 0 \} \subseteq \emptyset \)
   d) \( \emptyset \subseteq \{ 0 \} \)
   e) \( \{ 0 \} \in \{ 0 \} \)
   f) \( \{ 0 \} \subseteq \{ 0 \} \)
   g) \( \{ \emptyset \} \subseteq \{ \emptyset \} \)

10. Determine whether these statements are true or false.
    a) \( \emptyset \in \{ \emptyset \} \)
    b) \( \emptyset \in \{ \emptyset, \{ \emptyset \} \} \)
    c) \( \{ \{ \emptyset \} \} \subseteq \{ \emptyset \} \)
    d) \( \{ \{ \emptyset \} \} \subseteq \{ \emptyset, \{ \emptyset \} \} \)
    e) \( \{ \emptyset \} \subseteq \{ \emptyset, \{ \emptyset \} \} \)
    f) \( \{ \{ \emptyset \} \} \subseteq \{ \emptyset, \{ \emptyset \} \} \)

11. Determine whether each of these statements is true or false.
    a) \( x \in \{ x \} \)
    b) \( \{ x \} \subseteq \{ x \} \)
    c) \( x \in \{ x \} \)
    d) \( \{ x \} \subseteq \{ x \} \)
    e) \( \emptyset \subseteq \{ x \} \)
    f) \( \emptyset \subseteq \{ x \} \)

12. Use a Venn diagram to illustrate the subset of odd integers in the set of all positive integers not exceeding 10.
13. Use a Venn diagram to illustrate the set of all months of the year whose names do not contain the letter $R$ in the set of all months of the year.

14. Use a Venn diagram to illustrate the relationship $A \subseteq B$ and $B \subseteq C$.

15. Use a Venn diagram to illustrate the relationships $A \subset B$ and $B \subset C$.

16. Use a Venn diagram to illustrate the relationships $A \subset B$ and $A \subset C$.

17. Suppose that $A$, $B$, and $C$ are sets such that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$.

18. Find two sets $A$ and $B$ such that $A \in B$ and $A \subseteq B$.

19. What is the cardinality of each of these sets?
   a) $\{a\}$
   b) $\{\{a\}\}$
   c) $\{a, \{a\}\}$
   d) $\{a, \{a\}, \{\{a\}\}\}$

20. What is the cardinality of each of these sets?
   a) $\emptyset$
   b) $\{\emptyset\}$
   c) $\emptyset, \{\emptyset\}$
   d) $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$

21. Find the power set of each of these sets, where $a$ and $b$ are distinct elements.
   a) $\{a\}$
   b) $\{a, b\}$
   c) $\emptyset, \{\emptyset\}$

22. Can you conclude that $A = B$ if $A$ and $B$ are two sets with the same power set?

23. How many elements does each of these sets have where $a$ and $b$ are distinct elements?
   a) $P(\{a, b, \{a, b\}\})$
   b) $P(\emptyset, \{a\}, \{\{a\}\})$
   c) $P(P(\emptyset))$

24. Determine whether each of these sets is the power set of a set, where $a$ and $b$ are distinct elements.
   a) $\emptyset$
   b) $\{\emptyset, \{a\}\}$
   c) $\emptyset, \{a\}, \{\emptyset, a\}$
   d) $\emptyset, \{a\}, \{b\}, \{a, b\}$

25. Prove that $P(A) \subseteq P(B)$ if and only if $A \subseteq B$.

26. Show that if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.

27. Let $A = \{a, b, c, d\}$ and $B = \{y, z\}$. Find
   a) $A \times B$
   b) $B \times A$.

28. What is the Cartesian product $A \times B$, where $A$ is the set of courses offered by the mathematics department at a university and $B$ is the set of mathematics professors at this university? Give an example of how this Cartesian product can be used.

29. What is the Cartesian product $A \times B \times C$, where $A$ is the set of all airlines and $B$ and $C$ are both the set of all cities in the United States? Give an example of how this Cartesian product can be used.

30. Suppose that $A \times B = \emptyset$, where $A$ and $B$ are sets. What can you conclude?

31. Let $A$ be a set. Show that $\emptyset \times A = A \times \emptyset = \emptyset$.

32. Let $A = \{a, b, c\}$, $B = \{x, y\}$, and $C = \{0, 1\}$. Find
   a) $A \times B \times C$
   b) $C \times B \times A$
   c) $C \times A \times B$
   d) $B \times A \times B$.

33. Find $A^2$ if
   a) $A = \{0, 1, 3\}$
   b) $A = \{1, 2, a, b\}$.

34. Find $A^3$ if
   a) $A = \{a\}$
   b) $A = \{0, a\}$.

35. How many different elements does $A \times B$ have if $A$ has $m$ elements and $B$ has $n$ elements?

36. How many different elements does $A \times B \times C$ have if $A$ has $m$ elements, $B$ has $n$ elements, and $C$ has $p$ elements?

37. How many different elements does $A^n$ have when $A$ has $m$ elements and $n$ is a positive integer?

38. Show that $A \times B \neq B \times A$, when $A$ and $B$ are nonempty, unless $A = B$.

39. Explain why $A \times B \times C$ and $(A \times B) \times C$ are not the same.

40. Explain why $(A \times B) \times (C \times D)$ and $A \times (B \times C) \times D$ are not the same.

41. Translate each of these quantifications into English and determine its truth value.
   a) $\forall x \in \mathbb{R} (x^2 \neq -1)$
   b) $\exists x \in \mathbb{Z} (x^2 = 2)$
   c) $\forall x \in \mathbb{Z} (x^2 > 0)$
   d) $\exists x \in \mathbb{R} (x^2 = x)$

42. Translate each of these quantifications into English and determine its truth value.
   a) $\exists x \in \mathbb{R} (x^3 = -1)$
   b) $\exists x \in \mathbb{Z} (x + 1 > x)$
   c) $\forall x \in \mathbb{Z} (x - 1 \in \mathbb{Z})$
   d) $\forall x \in \mathbb{Z} (x^2 \in \mathbb{Z})$

43. Find the truth set of each of these predicates where the domain is the set of integers.
   a) $P(x): x^2 < 3$
   b) $Q(x): x^2 > x$
   c) $R(x): 2x + 1 = 0$

44. Find the truth set of each of these predicates where the domain is the set of integers.
   a) $P(x): x^2 \geq 1$
   b) $Q(x): x^2 = 2$
   c) $R(x): x < x^2$

45. The defining property of an ordered pair is that two ordered pairs are equal if and only if their first elements are equal and their second elements are equal. Surprisingly, instead of taking the ordered pair as a primitive concept, we can construct ordered pairs using basic notions from set theory. Show that if we define the ordered pair $(a, b)$ to be $\{(a), \{a, b\}\}$, then $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. (Hint: First show that $\{(a), \{a, b\}\} = \{(c), \{c, d\}\}$ if and only if $a = c$ and $b = d$.)

46. This exercise presents Russell’s paradox. Let $S$ be the set that contains a set $x$ if the set $x$ does not belong to itself, so that $S = \{x \mid x \notin x\}$.
   a) Show the assumption that $S$ is a member of $S$ leads to a contradiction.
   b) Show the assumption that $S$ is not a member of $S$ leads to a contradiction.

   By parts (a) and (b) it follows that the set $S$ cannot be defined as it was. This paradox can be avoided by restricting the types of elements that sets can have.

47. Describe a procedure for listing all the subsets of a finite set.
Exercises

1. Let $A$ be the set of students who live within one mile of school and let $B$ be the set of students who walk to classes. Describe the students in each of these sets.
   a) $A \cap B$
   b) $A \cup B$
   c) $A - B$
   d) $B - A$

2. Suppose that $A$ is the set of sophomores at your school and $B$ is the set of students in discrete mathematics at your school. Express each of these sets in terms of $A$ and $B$.
   a) the set of sophomores taking discrete mathematics in your school
   b) the set of sophomores at your school who are not taking discrete mathematics
   c) the set of students at your school who either are sophomores or are taking discrete mathematics
   d) the set of students at your school who either are not sophomores or are not taking discrete mathematics

3. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find
   a) $A \cup B$
   b) $A \cap B$
   c) $A - B$
   d) $B - A$

4. Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Find
   a) $A \cup B$
   b) $A \cap B$
   c) $A - B$
   d) $B - A$

In Exercises 5–10 assume that $A$ is a subset of some underlying universal set $U$.

5. Prove the complementation law in Table 1 by showing that $\overline{\overline{A}} = A$.

6. Prove the identity laws in Table 1 by showing that
   a) $A \cup \emptyset = A$
   b) $A \cup U = A$

7. Prove the domination laws in Table 1 by showing that
   a) $A \cup U = U$
   b) $A \cap \emptyset = \emptyset$

8. Prove the idempotent laws in Table 1 by showing that
   a) $A \cup A = A$
   b) $A \cap A = A$

9. Prove the complement laws in Table 1 by showing that
   a) $A \cup \overline{A} = U$
   b) $A \cap \overline{A} = \emptyset$

10. Show that
    a) $A - \emptyset = A$
    b) $\emptyset - A = \emptyset$

11. Let $A$ and $B$ be sets. Prove the commutative laws from Table 1 by showing that
    a) $A \cup B = B \cup A$
    b) $A \cap B = B \cap A$

12. Prove the first absorption law from Table 1 by showing that if $A$ and $B$ are sets, then $A \cup (A \cap B) = A$.

13. Prove the second absorption law from Table 1 by showing that if $A$ and $B$ are sets, then $A \cap (A \cup B) = A$.

14. Find the sets $A$ and $B$ if $A - B = \{1, 5, 7, 8\}$, $B - A = \{2, 10\}$, and $A \cap B = \{3, 6, 9\}$.

15. Prove the second De Morgan law in Table 1 by showing that if $A$ and $B$ are sets, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$
    a) by showing each side is a subset of the other side.
    b) using a membership table.

16. Let $A$ and $B$ be sets. Show that
    a) $(A \cap B) \subseteq A$
    b) $A \subseteq (A \cup B)$
    c) $A - B \subseteq A$
    d) $A \cap (B - A) = \emptyset$

17. Show that if $A$, $B$, and $C$ are sets, then $A \cap B \cap C = A \cap B \cap C$
    a) by showing each side is a subset of the other side.
    b) using a membership table.

18. Let $A$, $B$, and $C$ be sets. Show that
    a) $(A \cup B) \subseteq (A \cup B \cup C)$
    b) $(A \cap B \cap C) \subseteq (A \cap B)$
    c) $(A - B) - C \subseteq A - C$
    d) $(A - C) \cap (C - B) = \emptyset$
    e) $(B - A) \cup (C - A) = (B \cup C) - A$

19. Show that if $A$ and $B$ are sets, then
    a) $A - B = A \cap \overline{B}$
    b) $(A \cap B) \cap (A \cap \overline{B}) = A$

20. Show that if $A$ and $B$ are sets with $A \subseteq B$, then
    a) $A \cup B = B$
    b) $A \cap B = A$

21. Prove the first associative law from Table 1 by showing that if $A$, $B$, and $C$ are sets, then $A \cup (B \cup C) = (A \cup B) \cup C$.

22. Prove the second associative law from Table 1 by showing that if $A$, $B$, and $C$ are sets, then $A \cap (B \cap C) = (A \cap B) \cap C$.

23. Prove the first distributive law from Table 1 by showing that if $A$, $B$, and $C$ are sets, then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

24. Let $A$, $B$, and $C$ be sets. Show that $(A - B) - C = (A - C) - (B - C)$.

25. Let $A = \{0, 2, 4, 6, 8, 10\}$, $B = \{0, 1, 2, 3, 4, 5, 6\}$, and $C = \{4, 5, 6, 7, 8, 9, 10\}$. Find
    a) $A \cap B \cap C$
    b) $A \cup B \cup C$
    c) $(A \cap B) \cap C$
    d) $(A \cap B) \cup C$

26. Draw the Venn diagrams for each of these combinations of the sets $A$, $B$, and $C$.
    a) $A \cap (B \cup C)$
    b) $\overline{A} \cap \overline{B} \cap \overline{C}$
    c) $(A - B) \cup (A - C) \cup (B - C)$

27. Draw the Venn diagrams for each of these combinations of the sets $A$, $B$, and $C$.
    a) $A \cap (B - C)$
    b) $(A \cap B) \cup (A \cap C)$
    c) $(A \cap B) \cup (A \cap C)$

28. Draw the Venn diagrams for each of these combinations of the sets $A$, $B$, and $C$.
    a) $(A \cap B) \cap (C \cap D)$
    b) $\overline{A} \cup \overline{B} \cap (A \cap D)$
    c) $(A - B) \cap (C \cap D)$

29. What can you say about the sets $A$ and $B$ if we know that
    a) $(A \cup B) = A$
    b) $A \cap B = A$
    c) $A - B = A$
    d) $A \cap B = B \cap A$
    e) $A - B = B - A$
30. Can you conclude that \( A = B \) if \( A, B, \) and \( C \) are sets such that
a) \( A \cup C = B \cup C \)?
b) \( A \cap C = B \cap C \)?
c) \( A \cup C = B \cup C \) and \( A \cap C = B \cap C \)?
31. Let \( A \) and \( B \) be subsets of a universal set \( U \). Show that
\( A \subseteq B \) if and only if \( B \subseteq A \).
The symmetric difference of \( A \) and \( B \), denoted by \( A \oplus B \), is the set containing those elements in either \( A \) or \( B \), but not in both \( A \) and \( B \).
32. Find the symmetric difference of \( \{1, 3, 5\} \) and \( \{1, 2, 3\} \).
33. Find the symmetric difference of the set of computer science majors at a school and the set of mathematics majors at this school.
34. Draw a Venn diagram for the symmetric difference of the sets \( A \) and \( B \).
35. Show that \( A \oplus B = (A \cup B) - (A \cap B) \).
36. Show that \( A \oplus B = (A - B) \cup (B - A) \).
37. Show that if \( A \) is a subset of a universal set \( U \), then
a) \( A \oplus A = \emptyset \).
b) \( A \oplus \emptyset = A \).
c) \( A \oplus U = A \).
d) \( A \oplus \overline{A} = U \).
38. Show that if \( A \) and \( B \) are sets, then
a) \( A \oplus B = B \oplus A \).
b) \( (A \oplus B) \oplus B = A \).
39. What can you say about the sets \( A \) and \( B \) if \( A \oplus B = A \)?
40. Determine whether the symmetric difference is associative; that is, if \( A, B, \) and \( C \) are sets, does it follow that \( A \oplus (B \oplus C) = (A \oplus B) \oplus C \)?
41. Suppose that \( A, B, \) and \( C \) are sets such that \( A \oplus C = B \oplus C \). Must it be the case that \( A = B \)?
42. If \( A, B, C, \) and \( D \) are sets, does it follow that \( (A \oplus B) \oplus (C \oplus D) = (A \oplus C) \oplus (B \oplus D) \)?
43. If \( A, B, C, \) and \( D \) are sets, does it follow that \( (A \oplus B) \oplus (C \oplus D) = (A \oplus D) \oplus (B \oplus C) \)?
44. Show that if \( A \) and \( B \) are finite sets, then \( A \cup B \) is a finite set.
45. Show that if \( A \) is an infinite set, then whenever \( B \) is a set, \( A \cup B \) is also an infinite set.
46. Show that if \( A, B, \) and \( C \) are sets, then
\[
|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.
\]
(This is a special case of the inclusion–exclusion principle, which will be studied in Chapter 8.)
47. Let \( A_i = \{1, 2, 3, \ldots, i\} \) for \( i = 1, 2, 3, \ldots \). Find
\[
\bigcup_{i=1}^{n} A_i.
\]
48. Let \( A_i = \{\ldots, -2, -1, 0, 1, \ldots, i\} \). Find
\[
\bigcup_{i=1}^{n} A_i.
\]
49. Let \( A_i \) be the set of all nonempty bit strings (that is, bit strings of length at least one) of length not exceeding \( i \).
Find
\[
\bigcup_{i=1}^{n} A_i.
\]
50. Find \( \bigcap_{i=1}^{\infty} A_i \) and \( \bigcup_{i=1}^{\infty} A_i \) if for every positive integer \( i \),
a) \( A_i = \{i, i + 1, i + 2, \ldots\} \).
b) \( A_i = \{0, i\} \).
c) \( A_i = (0, i) \), that is, the set of real numbers \( x \) with \( 0 < x < i \).
d) \( A_i = (i, \infty) \), that is, the set of real numbers \( x \) with \( x > i \).
51. Find \( \bigcup_{i=1}^{\infty} A_i \) and \( \bigcap_{i=1}^{\infty} A_i \) if for every positive integer \( i \),
a) \( A_i = [-i, -i + 1, 0, 1, \ldots, i - 1, i] \).
b) \( A_i = [-i, i] \).
c) \( A_i = [-i, i] \), that is, the set of real numbers \( x \) with \( -i \leq x \leq i \).
d) \( A_i = [i, \infty) \), that is, the set of real numbers \( x \) with \( x \geq i \).
52. Suppose that the universal set is \( U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \). Express each of these sets with bit strings where the \( i \)th bit in the string is 1 if \( i \) is in the set and 0 otherwise.
a) \( \{3, 4, 5\} \)
b) \( \{1, 3, 6, 10\} \)
c) \( \{2, 3, 4, 7, 8, 9\} \)
53. Using the same universal set as in the last problem, find the set specified by each of these bit strings.
a) \( 111001111 \)
b) \( 0101111000 \)
c) \( 1000000001 \)
54. What subsets of a finite universal set do these bit strings represent?
a) the string with all zeros
b) the string with all ones
55. What is the bit string corresponding to the difference of two sets?
56. What is the bit string corresponding to the symmetric difference of two sets?
57. Show how bitwise operations on bit strings can be used to find these combinations of \( A = \{a, b, c, d, e\} \), \( B = \{b, c, d, g, p, i, v\} \), \( C = \{c, e, i, o, u, x, y, z\} \), and \( D = \{d, e, h, i, n, o, r, t, u, x, y\} \).
a) \( A \cup B \)
b) \( A \cap B \)
c) \( (A \cup D) \cap (B \cup C) \)
d) \( A \cup B \cup C \cup D \)
58. How can the union and intersection of \( n \) sets that all are subsets of the universal set \( U \) be found using bit strings?
The successor of the set \( A \) is the set \( A \cup \{A\} \).
59. Find the successors of the following sets.
a) \( \{1, 2, 3\} \)
b) \( \emptyset \)
c) \( \emptyset \)
d) \( \{\emptyset, \{\emptyset\}\} \)
Partial Functions

A program designed to evaluate a function may not produce the correct value of the function for all elements in the domain of this function. For example, a program may not produce a correct value because evaluating the function may lead to an infinite loop or an overflow. Similarly, in abstract mathematics, we often want to discuss functions that are defined only for a subset of the real numbers, such as $1/x$, $\sqrt{x}$, and $\arcsin(x)$. Also, we may want to use such notions as the "youngest child" function, which is undefined for a couple having no children, or the "time of sunrise," which is undefined for some days above the Arctic Circle. To study such situations, we use the concept of a partial function.

**DEFINITION 13**

A partial function $f$ from a set $A$ to a set $B$ is an assignment to each element $a$ in a subset of $A$, called the domain of definition of $f$, of a unique element $b$ in $B$. The sets $A$ and $B$ are called the domain and codomain of $f$, respectively. We say that $f$ is undefined for elements in $A$ that are not in the domain of definition of $f$. When the domain of definition of $f$ equals $A$, we say that $f$ is a total function.

**Remark:** We write $f : A \to B$ to denote that $f$ is a partial function from $A$ to $B$. Note that this is the same notation as is used for functions. The context in which the notation is used determines whether $f$ is a partial function or a total function.

**EXAMPLE 32** The function $f : \mathbb{Z} \to \mathbb{R}$ where $f(n) = \sqrt{n}$ is a partial function from $\mathbb{Z}$ to $\mathbb{R}$ where the domain of definition is the set of nonnegative integers. Note that $f$ is undefined for negative integers.

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**Exercises**

1. Why is $f$ not a function from $\mathbb{R}$ to $\mathbb{R}$ if
   a) $f(x) = 1/x$?
   b) $f(x) = \sqrt{x}$?
   c) $f(x) = \pm\sqrt{(x^2 + 1)}$?

2. Determine whether $f$ is a function from $\mathbb{Z}$ to $\mathbb{R}$ if
   a) $f(n) = \pm n$.
   b) $f(n) = \sqrt{n^2 + 1}$.
   c) $f(n) = 1/(n^2 - 4)$.

3. Determine whether $f$ is a function from the set of all bit strings to the set of integers if
   a) $f(S)$ is the position of a 0 bit in $S$.
   b) $f(S)$ is the number of 1 bits in $S$.
   c) $f(S)$ is the smallest integer $i$ such that the $i$th bit of $S$ is 1 and $f(S) = 0$ when $S$ is the empty string, the string with no bits.

4. Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.
   a) the function that assigns to each nonnegative integer its last digit
   b) the function that assigns the next largest integer to a positive integer
   c) the function that assigns to a bit string the number of one bits in the string
   d) the function that assigns to a bit string the number of bits in the string

5. Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.
   a) the function that assigns to each bit string the number of ones in the string minus the number of zeros in the string
   b) the function that assigns to each bit string twice the number of zeros in that string
   c) the function that assigns the number of bits left over when a bit string is split into bytes (which are blocks of 8 bits)
   d) the function that assigns to each positive integer the largest perfect square not exceeding this integer

6. Find the domain and range of these functions.
   a) the function that assigns to each pair of positive integers the first integer of the pair
   b) the function that assigns to each positive integer its largest decimal digit
   c) the function that assigns to a bit string the number of ones minus the number of zeros in the string
   d) the function that assigns to each positive integer the largest integer not exceeding the square root of the integer
   e) the function that assigns to a bit string the longest string of ones in the string
7. Find the domain and range of these functions.
   a) the function that assigns to each pair of positive integers the maximum of these two integers
   b) the function that assigns to each positive integer the number of digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 that do not appear as decimal digits of the integer
   c) the function that assigns to a bit string the number of times the block 11 appears
   d) the function that assigns to a bit string the numerical position of the first 1 in the string and that assigns the value 0 to a bit string consisting of all 0s

8. Find these values.
   a) \([1.1]\)
   b) \([-0.1]\)
   c) \([2.99]\)
   d) \([-2.99]\)
   e) \(\left[\frac{1}{2} + \left[\frac{1}{2}\right]\right]\)
   f) \(\left[\frac{1}{2}\right] + \left[\frac{1}{2}\right] + \frac{1}{2}\)

9. Find these values.
   a) \(\left[\frac{3}{4}\right]\)
   b) \(\left[\frac{-3}{4}\right]\)
   c) \(\left[-\frac{3}{4}\right]\)
   d) \(\left[-\frac{3}{4}\right]\)
   e) \(\left[\frac{3}{4}\right]\)
   f) \(\left[-\frac{3}{4}\right]\)
   g) \(\left[\frac{1}{2} + \left[\frac{1}{2}\right]\right]\)
   h) \(\left[\frac{1}{2} - \left[\frac{1}{2}\right]\right]\)

10. Determine whether each of these functions from \([a, b, c, d]\) to itself is one-to-one.
   a) \(f(a) = b, f(b) = a, f(c) = c, f(d) = d\)
   b) \(f(a) = b, f(b) = a, f(c) = d, f(d) = c\)
   c) \(f(a) = d, f(b) = b, f(c) = c, f(d) = d\)

11. Which functions in Exercise 10 are onto?

12. Determine whether each of these functions from \(Z\) to \(Z\) is one-to-one.
   a) \(f(n) = n - 1\)
   b) \(f(n) = n^2 + 1\)
   c) \(f(n) = n^3\)
   d) \(f(n) = \lfloor n/2\rfloor\)

13. Which functions in Exercise 12 are onto?

14. Determine whether \(f: Z \times Z \to Z\) is onto if
   a) \(f(m, n) = 2m - n\)
   b) \(f(m, n) = m^2 - n^2\)
   c) \(f(m, n) = m + n + 1\)
   d) \(f(m, n) = |m| - |n|\)
   e) \(f(m, n) = m^2 - 4\)

15. Determine whether the function \(f: Z \times Z \to Z\) is onto if
   a) \(f(m, n) = m + n\)
   b) \(f(m, n) = m^2 + n^2\)
   c) \(f(m, n) = m\)
   d) \(f(m, n) = |n|\)
   e) \(f(m, n) = m - n\)

16. Consider these functions from the set of students in a discrete mathematics class. Under what conditions is the function one-to-one if it assigns to a student his or her
   a) mobile phone number.
   b) student identification number.
   c) final grade in the class.
   d) home town.

17. Consider these functions from the set of teachers in a school. Under what conditions is the function one-to-one if it assigns to a teacher his or her
   a) office.
   b) assigned bus to chaperone in a group of buses taking students on a field trip.
   c) salary.
   d) social security number.

18. Specify a codomain for each of the functions in Exercise 16. Under what conditions is each of these functions with the codomain you specified onto?

19. Specify a codomain for each of the functions in Exercise 17. Under what conditions is each of the functions with the codomain you specified onto?

20. Give an example of a function from \(N\) to \(N\) that is
   a) one-to-one but not onto.
   b) onto but not one-to-one.
   c) both onto and one-to-one (but different from the identity function).
   d) neither one-to-one nor onto.

21. Give an explicit formula for a function from the set of integers to the set of positive integers that is
   a) one-to-one, but not onto.
   b) onto, but not one-to-one.
   c) one-to-one and onto.
   d) neither one-to-one nor onto.

22. Determine whether each of these functions is a bijection from \(R\) to \(R\).
   a) \(f(x) = -3x + 4\)
   b) \(f(x) = -3x^2 + 7\)
   c) \(f(x) = (x + 1)/(x + 2)\)
   d) \(f(x) = x^5 + 1\)

23. Determine whether each of these functions is a bijection from \(R\) to \(R\).
   a) \(f(x) = 2x + 1\)
   b) \(f(x) = x^3 + 1\)
   c) \(f(x) = x^3\)
   d) \(f(x) = (x^2 + 1)/(x^2 + 2)\)

24. Let \(f: R \to R\) and let \(f(x) > 0\) for all \(x \in R\). Show that \(f(x)\) is strictly increasing if and only if the function \(g(x) = 1/f(x)\) is strictly decreasing.

25. Let \(f: R \to R\) and let \(f(x) > 0\) for all \(x \in R\). Show that \(f(x)\) is strictly decreasing if and only if the function \(g(x) = 1/f(x)\) is strictly increasing.

26. a) Prove that a strictly increasing function from \(R\) to itself is one-to-one.
   b) Give an example of an increasing function from \(R\) to itself that is not one-to-one.

27. a) Prove that a strictly decreasing function from \(R\) to itself is one-to-one.
   b) Give an example of a decreasing function from \(R\) to itself that is not one-to-one.

28. Show that the function \(f(x) = e^x\) from the set of real numbers to the set of real numbers is not invertible, but if the codomain is restricted to the set of positive real numbers, the resulting function is invertible.
SOME INFINITE SERIES Although most of the summations in this book are finite sums, infinite series are important in some parts of discrete mathematics. Infinite series are usually studied in a course in calculus and even the definition of these series requires the use of calculus, but sometimes they arise in discrete mathematics, because discrete mathematics deals with infinite collections of discrete elements. In particular, in our future studies in discrete mathematics, we will find the closed forms for the infinite series in Examples 24 and 25 to be quite useful.

EXAMPLE 24 (Requires calculus) Let \( x \) be a real number with \( |x| < 1 \). Find \( \sum_{n=0}^{\infty} x^n \).

Solution: By Theorem 1 with \( a = 1 \) and \( r = x \) we see that \( \sum_{n=0}^{\infty} x^n = \frac{x^{k+1} - 1}{x - 1} \). Because \( |x| < 1 \), \( x^{k+1} \) approaches 0 as \( k \) approaches infinity. It follows that

\[
\sum_{n=0}^{\infty} x^n = \lim_{k \to \infty} \frac{x^{k+1} - 1}{x - 1} = \frac{0 - 1}{x - 1} = \frac{1}{1 - x}.
\]

We can produce new summation formulae by differentiating or integrating existing formulae.

EXAMPLE 25 (Requires calculus) Differentiating both sides of the equation

\[
\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x},
\]

from Example 24 we find that

\[
\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1 - x)^2}.
\]

(This differentiation is valid for \(|x| < 1\) by a theorem about infinite series.)

Exercises

1. Find these terms of the sequence \( \{a_n\} \), where \( a_n = 2 \cdot (-3)^n + 5^n \).
   a) \( a_0 \)  
   b) \( a_1 \)  
   c) \( a_4 \)  
   d) \( a_5 \)

2. What is the term \( a_8 \) of the sequence \( \{a_n\} \) if \( a_n \) equals
   a) \( 2^{n-1} \)  
   b) \( ? \)  
   c) \( 1 + (-1)^n \)  
   d) \( -(-2)^n \)

3. What are the terms \( a_0, a_1, a_2, \) and \( a_3 \) of the sequence \( \{a_n\} \), where \( a_n \) equals
   a) \( 2^n + 1 \)  
   b) \( (n + 1)^{n+1} \)  
   c) \( \lfloor n/2 \rfloor \)  
   d) \( \lceil n/2 \rceil + \lfloor n/2 \rfloor \)

4. What are the terms \( a_0, a_1, a_2, \) and \( a_3 \) of the sequence \( \{a_n\} \), where \( a_n \) equals
   a) \( (-2)^{n} \)  
   b) \( 3^n \)  
   c) \( 7 + 4^n \)  
   d) \( 2^n + (-2)^n \)

5. List the first 10 terms of each of these sequences.
   a) the sequence that begins with 2 and in which each successive term is 3 more than the preceding term
   b) the sequence that lists each positive integer three times, in increasing order
   c) the sequence that lists the odd positive integers in increasing order, listing each odd integer twice
   d) the sequence whose \( n \)th term is \( n! - 2^n \)
   e) the sequence that begins with 3, where each succeeding term is twice the preceding term
   f) the sequence whose first term is 2, second term is 4, and each succeeding term is the sum of the two preceding terms
   g) the sequence whose \( n \)th term is the number of bits in the binary expansion of the number \( n \) (defined in Section 4.2)
   h) the sequence where the \( n \)th term is the number of letters in the English word for the index \( n \)

6. List the first 10 terms of each of these sequences.
   a) the sequence obtained by starting with 10 and obtaining each term by subtracting 3 from the previous term
   b) the sequence whose \( n \)th term is the sum of the first \( n \) positive integers
   c) the sequence whose \( n \)th term is \( 3^n - 2^n \)
   d) the sequence whose \( n \)th term is \( \lfloor \sqrt{n} \rfloor \)
   e) the sequence whose first two terms are 1 and 5 and each succeeding term is the sum of the two previous terms
f) the sequence whose $n$th term is the largest integer whose binary expansion (defined in Section 4.2) has $n$ bits (Write your answer in decimal notation.)
g) the sequence whose terms are constructed sequentially as follows: start with 1, then add 1, then multiply by 1, then add 2, then multiply by 2, and so on
h) the sequence whose $n$th term is the largest integer $k$ such that $k! \leq n$

7. Find at least three different sequences beginning with the terms 1, 2, 4 whose terms are generated by a simple formula or rule.

8. Find at least three different sequences beginning with the terms 3, 5, 7 whose terms are generated by a simple formula or rule.

9. Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions.
   a) $a_n = 6a_{n-1}$, $a_0 = 2$
   b) $a_n = a_{n-1}^2$, $a_1 = 2$
   c) $a_n = a_{n-1} + 3a_{n-2}$, $a_0 = 1$, $a_1 = 2$
   d) $a_n = na_{n-1} + n^2a_{n-2}$, $a_0 = 1$, $a_1 = 1$
   e) $a_n = a_{n-1} + a_{n-3}$, $a_0 = 1$, $a_1 = 2$, $a_2 = 0$

10. Find the first six terms of the sequence defined by each of these recurrence relations and initial conditions.
   a) $a_n = -2a_{n-1}$, $a_0 = -1$
   b) $a_n = a_{n-1} - a_{n-2}$, $a_0 = 2$, $a_1 = -1$
   c) $a_n = 3a_{n-1}$, $a_0 = 1$
   d) $a_n = na_{n-1} + a_{n-2}^2$, $a_0 = -1$, $a_1 = 0$
   e) $a_n = a_{n-1} - a_{n-2} + a_{n-3}$, $a_0 = 1$, $a_1 = 1$, $a_2 = 2$

11. Let $a_n = 2^n + 5 \cdot 3^n$ for $n = 0, 1, 2, \ldots$
   a) Find $a_0$, $a_1$, $a_2$, $a_3$, and $a_4$.
   b) Show that $a_2 = 5a_1 - 6a_0$, $a_3 = 5a_2 - 6a_1$, and $a_4 = 5a_3 - 6a_2$.
   c) Show that $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers $n$ with $n \geq 2$.

12. Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$ if
   a) $a_0 = 0$
   b) $a_1 = 1$
   c) $a_0 = (-4)^n$
   d) $a_1 = 2(-4)^n + 3$

13. Is the sequence $\{a_n\}$ a solution of the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$ if
   a) $a_0 = 0$
   b) $a_0 = 1$
   c) $a_0 = 2^n$
   d) $a_0 = 4^n$
   e) $a_0 = n^4$
   f) $a_0 = 2 \cdot 4^n + 3n^4$
   g) $a_0 = (-4)^n$
   h) $a_0 = n^24^n$

14. For each of these sequences find a recurrence relation satisfied by the sequence. (The answers are not unique because there are infinitely many different recurrence relations satisfied by any sequence.)
   a) $a_n = 3$
   b) $a_n = 2n$
   c) $a_n = 2n + 3$
   d) $a_n = 5^n$
   e) $a_n = n^2$
   f) $a_n = n^2 + n$
   g) $a_n = n + (-1)^n$
   h) $a_n = n!$

15. Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$ if
   a) $a_n = -n + 2$
   b) $a_n = 5(-1)^n - n + 2$
   c) $a_n = 3(-1)^n + 2^n - n + 2$
   d) $a_n = 7 \cdot 2^n - n + 2$

16. Find the solution to each of these recurrence relations with the given initial conditions. Use an iterative approach such as that used in Example 10.
   a) $a_n = -a_{n-1}$, $a_0 = 5$
   b) $a_n = a_{n-1} + 3$, $a_0 = 1$
   c) $a_n = a_{n-1} - n$, $a_0 = 4$
   d) $a_n = 2a_{n-1} - 3$, $a_0 = -1$
   e) $a_n = (n + 1)a_{n-1}$, $a_0 = 2$
   f) $a_n = 2an_{n-1}$, $a_0 = 3$
   g) $a_n = a_{n-1} + n - 1$, $a_0 = 7$

17. Find the solution to each of these recurrence relations and initial conditions. Use an iterative approach such as that used in Example 10.
   a) $a_n = 3a_{n-1}$, $a_0 = 2$
   b) $a_n = a_{n-1} + 2$,$a_0 = 3$
   c) $a_n = a_{n-1} + n$, $a_0 = 1$
   d) $a_n = a_{n-1} + 2n + 3$, $a_0 = 4$
   e) $a_n = 2a_{n-1} - 1$, $a_0 = 1$
   f) $a_n = 3a_{n-1} + 1$, $a_0 = 1$
   g) $a_n = na_{n-1}$, $a_0 = 5$
   h) $a_n = 2a_{n-1}$, $a_0 = 1$

18. A person deposits $1000 in an account that yields 9% interest compounded annually.
   a) Set up a recurrence relation for the amount in the account at the end of $n$ years.
   b) Find an explicit formula for the amount in the account at the end of $n$ years.
   c) How much money will the account contain after 100 years?

19. Suppose that the number of bacteria in a colony triples every hour.
   a) Set up a recurrence relation for the number of bacteria after $n$ hours have elapsed.
   b) If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?

20. Assume that the population of the world in 2010 was 6.9 billion and is growing at the rate of 1.1% a year.
   a) Set up a recurrence relation for the population of the world $n$ years after 2010.
   b) Find an explicit formula for the population of the world $n$ years after 2010.
   c) What will the population of the world be in 2030?

21. A factory makes custom sports cars at an increasing rate. In the first month only one car is made, in the second month two cars are made, and so on, with $n$ cars made in the $n$th month.
   a) Set up a recurrence relation for the number of cars produced in the first $n$ months by this factory.
   b) How many cars are produced in the first year?
   c) Find an explicit formula for the number of cars produced in the first $n$ months by this factory.

22. An employee joined a company in 2009 with a starting salary of $50,000. Every year this employee receives a raise of $1000 plus 5% of the salary of the previous year.
2.4 Sequences and Summations

30. What are the values of these sums, where $S = \{1, 3, 5, 7\}$?
   a) $\sum_{j \in S} j$
   b) $\sum_{j \in S} j^2$
   c) $\sum_{j \in S} \frac{1}{j}$
   d) $\sum_{j \in S} 1$

31. What is the value of each of these sums of terms of a geometric progression?
   a) $\sum_{j=0}^{8} 3 \cdot 2^j$
   b) $\sum_{j=0}^{8} 2^j$
   c) $\sum_{j=2}^{8} (-3)^j$
   d) $\sum_{j=0}^{8} 2 \cdot (-3)^j$

32. Find the value of each of these sums.
   a) $\sum_{j=0}^{8} (1 + (-1)^j)$
   b) $\sum_{j=0}^{8} (3^j - 2^j)$
   c) $\sum_{j=0}^{8} (2 \cdot 3^j + 2^j)$
   d) $\sum_{j=0}^{8} (2^j + 1 - 2^j)$

33. Compute each of these double sums.
   a) $\sum_{i=1}^{2} \sum_{j=1}^{3} (i + j)$
   b) $\sum_{i=0}^{3} \sum_{j=0}^{3} (2i + 3j)$
   c) $\sum_{i=1}^{2} \sum_{j=0}^{i} i$
   d) $\sum_{i=0}^{2} \sum_{j=0}^{i} ij$

34. Compute each of these double sums.
   a) $\sum_{i=1}^{2} \sum_{j=1}^{2} (i - j)$
   b) $\sum_{i=0}^{2} \sum_{j=0}^{3} (3i + 2j)$
   c) $\sum_{i=1}^{3} \sum_{j=0}^{2} j$
   d) $\sum_{i=0}^{3} \sum_{j=0}^{3} j^2$

35. Show that $\sum_{i=1}^{n} (a_j - a_{j-1}) = a_n - a_0$, where $a_0, a_1, \ldots, a_n$ is a sequence of real numbers. This type of sum is called telescoping.

36. Use the identity $1/k(k + 1) = 1/k - 1/(k + 1)$ and Exercise 35 to compute $\sum_{k=1}^{n} 1/k(k + 1)$.

37. Sum both sides of the identity $k^2 - (k - 1)^2 = 2k - 1$ from $k = 1$ to $k = n$ and use Exercise 35 to find a formula for $\sum_{k=1}^{n} (2k - 1)$ (the sum of the first $n$ odd natural numbers).

38. Use the technique given in Exercise 35, together with the result of Exercise 37b, to derive the formula for $\sum_{k=1}^{n} k^2$ given in Table 2. [Hint: Take $a_k = k^3$ in the telescoping sum in Exercise 35.]

39. Find $\sum_{k=0}^{200} k$. (Use Table 2.)

40. Find $\sum_{k=0}^{200} k^3$. (Use Table 2.)

41. Find a formula for $\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor$, when $m$ is a positive integer.

42. Find a formula for $\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor$, when $m$ is a positive integer.

There is also a special notation for products. The product of $a_m, a_{m+1}, \ldots, a_n$ is represented by $\prod_{j=m}^{n} a_j$, read as the product from $j = m$ to $j = n$ of $a_j$. 

---

27. Show that if $a_n$ denotes the $n$th positive integer that is not a perfect square, then $a_n = n + \lfloor \sqrt{n} \rfloor$, where $|x|$ denotes the integer closest to the real number $x$.

28. Let $a_n$ be the $n$th term of the sequence $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 6, 6, 6, 6, 6, \ldots$, constructed by including the integer $k$ exactly $k$ times. Show that $a_n = \lfloor \sqrt{2n + 1/2} \rfloor$. 

29. What are the values of these sums?
   a) $\sum_{k=1}^{10} (k + 1)$
   b) $\sum_{j=0}^{8} (-2)^j$
   c) $\sum_{i=1}^{10} 3$
   d) $\sum_{j=0}^{8} (2^{j+1} - 2^j)$
FIGURE 2 Showing that the Halting Problem is Unsolvable.

is "halt," then by the definition of $K$ we see that $K(K)$ loops forever, in violation of what $H$ tells us. In both cases, we have a contradiction.

Thus, $H$ cannot always give the correct answers. Consequently, there is no procedure that solves the halting problem.

Exercises

1. List all the steps used by Algorithm 1 to find the maximum of the list 1, 8, 12, 9, 11, 2, 14, 5, 10, 4.

2. Determine which characteristics of an algorithm described in the text (after Algorithm 1) the following procedures have and which they lack.
   a) procedure double($n$: positive integer)
      while $n > 0$
      $n := 2n$
   b) procedure divide($n$: positive integer)
      while $n ≥ 0$
      $m := 1/n$
      $n := n - 1$
   c) procedure sum($n$: positive integer)
      $sum := 0$
      while $i < 10$
      $sum := sum + i$
   d) procedure choose($a, b$: integers)
      $x :=$ either $a$ or $b$

3. Devise an algorithm that finds the sum of all the integers in a list.

4. Describe an algorithm that takes as input a list of $n$ integers and produces as output the largest difference obtained by subtracting an integer in the list from the one following it.

5. Describe an algorithm that takes as input a list of $n$ integers in nondecreasing order and produces the list of all values that occur more than once. (Recall that a list of integers is nondecreasing if each integer in the list is at least as large as the previous integer in the list.)

6. Describe an algorithm that takes as input a list of $n$ integers and finds the number of negative integers in the list.

7. Describe an algorithm that takes as input a list of $n$ integers and finds the location of the last even integer in the list or returns 0 if there are no even integers in the list.

8. Describe an algorithm that takes as input a list of $n$ distinct integers and finds the location of the largest even integer in the list or returns 0 if there are no even integers in the list.

9. A palindrome is a string that reads the same forward and backward. Describe an algorithm for determining whether a string of $n$ characters is a palindrome.

10. Devise an algorithm to compute $x^n$, where $x$ is a real number and $n$ is an integer. [Hint: First give a procedure for computing $x^n$ when $n$ is nonnegative by successive multiplication by $x$, starting with 1. Then extend this procedure, and use the fact that $x^{-n} = 1/x^n$ to compute $x^n$ when $n$ is negative.]

11. Describe an algorithm that interchanges the values of the variables $x$ and $y$, using only assignments. What is the minimum number of assignment statements needed to do this?

12. Describe an algorithm that uses only assignment statements that replaces the triple $(x, y, z)$ with $(y, z, x)$. What is the minimum number of assignment statements needed?

13. List all the steps used to search for 9 in the sequence 1, 3, 4, 5, 6, 8, 9, 11 using
   a) a linear search.
   b) a binary search.

14. List all the steps used to search for 7 in the sequence given in Exercise 13 for both a linear search and a binary search.

15. Describe an algorithm that inserts an integer $x$ in the appropriate position into the list $a_1, a_2, \ldots, a_n$ of integers that are in increasing order.

16. Describe an algorithm for finding the smallest integer in a finite sequence of natural numbers.

17. Describe an algorithm that locates the first occurrence of the largest element in a finite list of integers, where the integers in the list are not necessarily distinct.

18. Describe an algorithm that locates the last occurrence of the smallest element in a finite list of integers, where the integers in the list are not necessarily distinct.