Problem 1.1.2 Solution

Based on the Venn diagram on the right, the answers are mostly fairly straightforward. The only trickiness is that a pizza is either Tuscan \( (T) \) or Neapolitan \( (N) \) so \( \{N,T\} \) is a partition but they are not depicted as a partition. Specifically, the event \( N \) is the region of the Venn diagram outside of the “square block” of event \( T \). If this is clear, the questions are easy.

(a) Since \( N = T^c \), \( N \cap M \neq \emptyset \). Thus \( N \) and \( M \) are not mutually exclusive.

(b) Every pizza is either Neapolitan \( (N) \), or Tuscan \( (T) \). Hence \( N \cup T = S \) so that \( N \) and \( T \) are collectively exhaustive. Thus its also (trivially) true that \( N \cup T \cup M = S \). That is, \( R \), \( T \) and \( M \) are also collectively exhaustive.

(c) From the Venn diagram, \( T \) and \( O \) are mutually exclusive. In words, this means that Tuscan pizzas never have onions or pizzas with onions are never Tuscan. As an aside, “Tuscan” is a fake pizza designation; one shouldn’t conclude that people from Tuscany actually dislike onions.

(d) From the Venn diagram, \( M \cap T \) and \( O \) are mutually exclusive. Thus Gerlanda’s doesn’t make Tuscan pizza with mushrooms and onions.

(e) Yes. In terms of the Venn diagram, these pizzas are in the set \( (T \cup M \cup O)^c \).

Problem 1.2.2 Solution

(a) The sample space of the experiment is

\[ S = \{aaa, aaf, afa, faa, ffa, faf, aff, fff\}. \tag{1} \]
(b) The event that the circuit from $Z$ fails is
\[ Z_F = \{aaf, aff, faf, fff\}. \] (2)

The event that the circuit from $X$ is acceptable is
\[ X_A = \{aaa, aaf, afa, aff\}. \] (3)

(c) Since $Z_F \cap X_A = \{aaf, aff\} \neq \emptyset$, $Z_F$ and $X_A$ are not mutually exclusive.

(d) Since $Z_F \cup X_A = \{aaa, aaf, afa, aff, faf, fff\} \neq S$, $Z_F$ and $X_A$ are not collectively exhaustive.

(e) The event that more than one circuit is acceptable is
\[ C = \{aaa, aaf, afa, faa\}. \] (4)

The event that at least two circuits fail is
\[ D = \{ffa, faf, aff, fff\}. \] (5)

(f) Inspection shows that $C \cap D = \emptyset$ so $C$ and $D$ are mutually exclusive.

(g) Since $C \cup D = S$, $C$ and $D$ are collectively exhaustive.

**Problem 1.3.5 Solution**

The sample space of the experiment is
\[ S = \{LF, BF, LW, BW\}. \] (1)

From the problem statement, we know that $P[LF] = 0.5$, $P[BF] = 0.2$ and $P[BW] = 0.2$. This implies $P[LW] = 1 - 0.5 - 0.2 - 0.2 = 0.1$. The questions can be answered using Theorem 1.5.

(a) The probability that a program is slow is
\[ P[W] = P[LW] + P[BW] = 0.1 + 0.2 = 0.3. \] (2)

(b) The probability that a program is big is
\[ P[B] = P[BF] + P[BW] = 0.2 + 0.2 = 0.4. \] (3)

(c) The probability that a program is slow or big is
\[ P[W \cup B] = P[W] + P[B] - P[BW] = 0.3 + 0.4 - 0.2 = 0.5. \] (4)
Problem 1.3.6 Solution
A sample outcome indicates whether the cell phone is handheld (H) or mobile (M) and whether the speed is fast (F) or slow (W). The sample space is

\[ S = \{HF, HW, MF, MW\}. \]  (1)

The problem statement tells us that \( P[HF] = 0.2 \), \( P[MW] = 0.1 \) and \( P[F] = 0.5 \). We can use these facts to find the probabilities of the other outcomes. In particular,

\[ P[F] = P[HF] + P[MF]. \]  (2)

This implies

\[ P[MF] = P[F] - P[HF] = 0.5 - 0.2 = 0.3. \]  (3)

Also, since the probabilities must sum to 1,

\[ = 1 - 0.2 - 0.3 - 0.1 = 0.4. \]  (4)

Now that we have found the probabilities of the outcomes, finding any other probability is easy.

(a) The probability a cell phone is slow is

\[ P[W] = P[HW] + P[MW] = 0.4 + 0.1 = 0.5. \]  (5)

(b) The probability that a cell phone is mobile and fast is \( P[MF] = 0.3 \).

(c) The probability that a cell phone is handheld is

\[ P[H] = P[HF] + P[HW] = 0.2 + 0.4 = 0.6. \]  (6)

Problem 1.3.10 Solution
Each statement is a consequence of part 4 of Theorem 1.4.

(a) Since \( A \subset A \cup B \), \( P[A] \leq P[A \cup B] \).

(b) Since \( B \subset A \cup B \), \( P[B] \leq P[A \cup B] \).

(c) Since \( A \cap B \subset A \), \( P[A \cap B] \leq P[A] \).

(d) Since \( A \cap B \subset B \), \( P[A \cap B] \leq P[B] \).
Problem 1.4.1 Solution

Each question requests a conditional probability.

(a) Note that the probability a call is brief is

\[ P[B] = P[H_0B] + P[H_1B] + P[H_2B] = 0.6. \] (1)

The probability a brief call will have no handoffs is

\[ P[H_0|B] = \frac{P[H_0B]}{P[B]} = \frac{0.4}{0.6} = \frac{2}{3}. \] (2)

(b) The probability of one handoff is \( P[H_1] = P[H_1B] + P[H_1L] = 0.2 \). The probability that a call with one handoff will be long is

\[ P[L|H_1] = \frac{P[H_1L]}{P[H_1]} = \frac{0.1}{0.2} = \frac{1}{2}. \] (3)

(c) The probability a call is long is \( P[L] = 1 - P[B] = 0.4 \). The probability that a long call will have one or more handoffs is

\[ P[H_1 \cup H_2|L] = \frac{P[H_1L \cup H_2L]}{P[L]} \]
\[ = \frac{P[H_1L] + P[H_2L]}{P[L]} = \frac{0.1 + 0.2}{0.4} = \frac{3}{4}. \] (4)

Problem 1.4.2 Solution

Let \( s_i \) denote the outcome that the roll is \( i \). So, for \( 1 \leq i \leq 6, R_i = \{s_i\} \). Similarly, \( G_j = \{s_{j+1}, \ldots, s_6\} \).

(a) Since \( G_1 = \{s_2, s_3, s_4, s_5, s_6\} \) and all outcomes have probability \( 1/6 \), \( P[G_1] = 5/6 \). The event \( R_3G_1 = \{s_3\} \) and \( P[R_3G_1] = 1/6 \) so that

\[ P[R_3|G_1] = \frac{P[R_3G_1]}{P[G_1]} = \frac{1}{5}. \] (1)
(b) The conditional probability that 6 is rolled given that the roll is greater than 3 is

$$P[R_6|G_3] = \frac{P[R_6G_3]}{P[G_3]} = \frac{P[s_6]}{P[s_4, s_5, s_6]} = \frac{1/6}{3/6} = \frac{1}{3}.$$  \hspace{1cm} (2)

(c) The event $E$ that the roll is even is $E = \{s_2, s_4, s_6\}$ and has probability $3/6$. The joint probability of $G_3$ and $E$ is

$$P[G_3E] = P[s_4, s_6] = 1/3.$$  \hspace{1cm} (3)

The conditional probabilities of $G_3$ given $E$ is

$$P[G_3|E] = \frac{P[G_3E]}{P[E]} = \frac{1/3}{1/2} = \frac{2}{3}.$$  \hspace{1cm} (4)

(d) The conditional probability that the roll is even given that it’s greater than 3 is

$$P[E|G_3] = \frac{P[EG_3]}{P[G_3]} = \frac{1/3}{1/2} = \frac{2}{3}.$$  \hspace{1cm} (5)

**Problem 1.4.7 Solution**

The sample outcomes can be written $ijk$ where the first card drawn is $i$, the second is $j$ and the third is $k$. The sample space is

$$S = \{234, 243, 324, 342, 423, 432\}.$$  \hspace{1cm} (1)

and each of the six outcomes has probability $1/6$. The events $E_1, E_2, E_3, O_1, O_2, O_3$ are

$$E_1 = \{234, 243, 423, 432\}, \quad O_1 = \{324, 342\}, \quad \hspace{1cm} (2)$$
$$E_2 = \{243, 324, 342, 423\}, \quad O_2 = \{234, 432\}, \quad \hspace{1cm} (3)$$
$$E_3 = \{234, 324, 342, 432\}, \quad O_3 = \{243, 423\}. \quad \hspace{1cm} (4)

(a) The conditional probability the second card is even given that the first card is even is

$$P[E_2|E_1] = \frac{P[E_2E_1]}{P[E_1]} = \frac{P[243, 423]}{P[234, 243, 423, 432]} = \frac{2/6}{4/6} = 1/2.$$  \hspace{1cm} (5)
(b) The conditional probability the first card is even given that the second card is even is
\[
P [E_1|E_2] = \frac{P [E_1E_2]}{P [E_2]} = \frac{P [243, 423]}{P [243, 324, 342, 423]} = \frac{2/6}{4/6} = 1/2. \tag{6}
\]

(c) The probability the first two cards are even given the third card is even is
\[
P [E_1E_2|E_3] = \frac{P [E_1E_2E_3]}{P [E_3]} = 0. \tag{7}
\]

(d) The conditional probabilities the second card is even given that the first card is odd is
\[
P [E_2|O_1] = \frac{P [O_1E_2]}{P [O_1]} = \frac{P [O_1]}{P [O_1]} = 1. \tag{8}
\]

(e) The conditional probability the second card is odd given that the first card is odd is
\[
P [O_2|O_1] = \frac{P [O_1O_2]}{P [O_1]} = 0. \tag{9}
\]

**Problem 1.5.1 Solution**

From the table we look to add all the mutually exclusive events to find each probability.

(a) The probability that a caller makes no hand-offs is
\[
P [H_0] = P [LH_0] + P [BH_0] = 0.1 + 0.4 = 0.5. \tag{1}
\]

(b) The probability that a call is brief is
\[
P [B] = P [BH_0] + P [BH_1] + P [BH_2] = 0.4 + 0.1 + 0.1 = 0.6. \tag{2}
\]

(c) The probability that a call is long or makes at least two hand-offs is
\[
P [L \cup H_2] = P [LH_0] + P [LH_1] + P [LH_2] + P [BH_2]
= 0.1 + 0.1 + 0.2 + 0.1 = 0.5. \tag{3}
\]
Problem 1.5.2 Solution

(a) From the given probability distribution of billed minutes, \( M \), the probability that a call is billed for more than 3 minutes is

\[
P[L] = 1 - P[3 \text{ or fewer billed minutes}]
= 1 - P[B_1] - P[B_2] - P[B_3]
= 1 - \alpha - \alpha(1 - \alpha) - \alpha(1 - \alpha)^2
= (1 - \alpha)^3 = 0.57. \tag{1}
\]

(b) The probability that a call will billed for 9 minutes or less is

\[
P[9 \text{ minutes or less}] = \sum_{i=1}^{9} \alpha(1 - \alpha)^{i-1} = 1 - (0.57)^3. \tag{2}
\]