Section 2.1

Tree Diagrams
Example 2.3 Problem

Suppose you have two coins, one biased, one fair, but you don’t know which coin is which. Coin 1 is biased. It comes up heads with probability 3/4, while coin 2 comes up heads with probability 1/2. Suppose you pick a coin at random and flip it. Let $C_i$ denote the event that coin $i$ is picked. Let $H$ and $T$ denote the possible outcomes of the flip. Given that the outcome of the flip is a head, what is $P[C_1|H]$, the probability that you picked the biased coin? Given that the outcome is a tail, what is the probability $P[C_1|T]$ that you picked the biased coin?
Example 2.3  Solution

First, we construct the sample tree on the left. To find the conditional probabilities, we see

\[ P[C_1|H] = \frac{P[C_1H]}{P[H]} = \frac{P[C_1H]}{P[C_1H] + P[C_2H]} \]

From the leaf probabilities in the sample tree,

\[ P[C_1|H] = \frac{3/8}{3/8 + 1/4} = \frac{3}{5}. \]

Similarly,

\[ P[C_1|T] = \frac{P[C_1T]}{P[T]} = \frac{P[C_1T]}{P[C_1T] + P[C_2T]} = \frac{1/8}{1/8 + 1/4} = \frac{1}{3}. \]  \( \text{(1)} \)

As we would expect, we are more likely to have chosen coin 1 when the first flip is heads, but we are more likely to have chosen coin 2 when the first flip is tails.
Section 2.2

Counting Methods
Theorem 2.1

An experiment consists of two subexperiments. If one subexperiment has $k$ outcomes and the other subexperiment has $n$ outcomes, then the experiment has $nk$ outcomes.
Theorem 2.2

The number of $k$-permutations of $n$ distinguishable objects is

$$(n)_k = n(n - 1)(n - 2) \cdots (n - k + 1) = \frac{n!}{(n-k)!}.$$
Problem 2.2.8

Consider a language containing four letters: $A$, $B$, $C$, $D$. How many three-letter words can you form in this language? How many four-letter words can you form if each letter appears only once in each word?
Theorem 2.3

The number of ways to choose $k$ objects out of $n$ distinguishable objects is

$$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}.$$
Definition 2.1 \( n \text{ choose } k \)

For an integer \( n \geq 0 \), we define

\[
\binom{n}{k} = \begin{cases} 
\frac{n!}{k!(n-k)!} & k = 0, 1, \ldots, n, \\
0 & \text{otherwise}.
\end{cases}
\]
Theorem 2.5

For $n$ repetitions of a subexperiment with sample space $S_{\text{sub}} = \{s_0, \ldots, s_{m-1}\}$, the sample space $S$ of the sequential experiment has $m^n$ outcomes.
Problem 2.2.7

Consider a binary code with 5 bits (0 or 1) in each code word. An example of a code word is 01010. How many different code words are there? How many code words have exactly three 0’s?
Example 2.17 Problem

For five subexperiments with sample space $S_{\text{Sub}} = \{0, 1\}$, what is the number of observation sequences in which 0 appears $n_0 = 2$ times and 1 appears $n_1 = 3$ times?
Quiz 2.2

Consider a binary code with 4 bits (0 or 1) in each code word. An example of a code word is 0110.

(a) How many different code words are there?
(b) How many code words have exactly two zeroes?
(c) How many code words begin with a zero?
(d) In a constant-ratio binary code, each code word has $N$ bits. In every word, $M$ of the $N$ bits are 1 and the other $N - M$ bits are 0. How many different code words are in the code with $N = 8$ and $M = 3$?
Problem 2.2.12

A basketball team has three pure centers, four pure forwards, four pure guards, and one swingman who can play either guard or forward. A pure position player can play only the designated position. If the coach must start a lineup with one center, two forwards, and two guards, how many possible lineups can the coach choose?
Theorem 2.7

For $n$ repetitions of a subexperiment with sample space $S = \{s_0, \ldots, s_{m-1}\}$, the number of length $n = n_0 + \cdots + n_{m-1}$ observation sequences with $s_i$ appearing $n_i$ times is

$$\binom{n}{n_0, \ldots, n_{m-1}} = \frac{n!}{n_0! n_1! \cdots n_{m-1}!}.$$
Definition 2.2 Multinomial Coefficient

For an integer $n \geq 0$, we define

$$\binom{n}{n_0, \ldots, n_{m-1}} = \begin{cases} \frac{n!}{n_0!n_1!\cdots n_{m-1}!} & n_0 + \cdots + n_{m-1} = n; \\ 0 & n_i \in \{0, 1, \ldots, n\}, i = 0, 1, \ldots, m-1, \\ otherwise. & \end{cases}$$
Example 2.16

There are ten students in a probability class. Each earns a grade \( s \in S_{\text{sub}} = \{A, B, C, F\} \). We use the notation \( x_i \) to denote the grade of the \( i \)th student. For example, the grades for the class could be

\[
x_1 x_2 \cdots x_{10} = CBBACFBACF
\]  \hspace{1cm} (1)

The sample space \( S \) of possible sequences contains \( 4^{10} = 1,048,576 \) outcomes.
Section 2.3

Independent Trials
Section 2.3

Independent Trials
Theorem 2.8

The probability of $n_0$ failures and $n_1$ successes in $n = n_0 + n_1$ independent trials is

$$P[E_{n_0,n_1}] = \binom{n}{n_1} (1 - p)^{n-n_1} p^{n_1} = \binom{n}{n_0} (1 - p)^{n_0} p^{n-n_0}.$$
Example 2.19 Problem

What is the probability $P[E_{2,3}]$ of two failures and three successes in five independent trials with success probability $p$. 
Problem 2.3.1

Consider a binary code with 5 bits (0 or 1) in each code word. An example of a code word is 01010. In each code word, a bit is a zero with probability 0.8, independent of any other bit.

(a) What is the probability of the code word 00111?
(b) What is the probability that a code word contains exactly three ones?
Example 2.21 Problem

To communicate one bit of information reliably, cellular phones transmit the same binary symbol five times. Thus the information “zero” is transmitted as 00000 and “one” is 11111. The receiver detects the correct information if three or more binary symbols are received correctly. What is the information error probability $P[E]$, if the binary symbol error probability is $q = 0.1$?
Theorem 2.9

A subexperiment has sample space $S_{\text{sub}} = \{s_0, \ldots, s_{m-1}\}$ with $P[s_i] = p_i$. For $n = n_0 + \cdots + n_{m-1}$ independent trials, the probability of $n_i$ occurrences of $s_i$, $i = 0, 1, \ldots, m-1$, is

$$P \left[ E_{n_0, \ldots, n_{m-1}} \right] = \binom{n}{n_0, \ldots, n_{m-1}} p_0^{n_0} \cdots p_{m-1}^{n_{m-1}}.$$
Example 2.22

A packet processed by an Internet router carries either audio information with probability $7/10$, video, with probability $2/10$, or text with probability $1/10$. Let $E_{a,v,t}$ denote the event that the router processes $a$ audio packets, $v$ video packets, and $t$ text packets in a sequence of 100 packets. In this case,

$$P[E_{a,v,t}] = \binom{100}{a,v,t} \left( \frac{7}{10} \right)^a \left( \frac{2}{10} \right)^v \left( \frac{1}{10} \right)^t$$

(1)

Keep in mind that by the extended definition of the multinomial coefficient, $P[E_{a,v,t}]$ is nonzero only if $a + v + t = 100$ and $a$, $v$, and $t$ are nonnegative integers.
Example 2.23 Problem

Continuing with Example 2.16, suppose in testing a microprocessor that all four grades have probability 0.25, independent of any other microprocessor. In testing \( n = 100 \) microprocessors, what is the probability of exactly 25 microprocessors of each grade?
Quiz 2.3

Data packets containing 100 bits are transmitted over a communication link. A transmitted bit is received in error (either a 0 sent is mistaken for a 1, or a 1 sent is mistaken for a 0) with probability $\epsilon = 0.01$, independent of the correctness of any other bit. The packet has been coded in such a way that if three or fewer bits are received in error, then those bits can be corrected. If more than three bits are received in error, then the packet is decoded with errors.

(a) Let $E_{k,100-k}$ denote the event that a received packet has $k$ bits in error and $100 - k$ correctly decoded bits. What is $P[E_{k,100-k}]$ for $k = 0, 1, 2, 3$?

(b) Let $C$ denote the event that a packet is decoded correctly. What is $P[C]$?
Section 2.4

Reliability Analysis
Quiz 2.4

A memory module consists of nine chips. The device is designed with redundancy so that it works even if one of its chips is defective. Each chip contains $n$ transistors and functions properly only if all of its transistors work. A transistor works with probability $p$ independent of any other transistor.

(a) What is the probability $P[C]$ that a chip works?

(b) What is the probability $P[M]$ that the memory module works?