Yates and Goodman 3e Solution Set: 4.2.1, 4.3.1, 4.3.4, 4.4.2, 4.4.4, 4.4.6, 4.5.6, 4.5.10, 4.5.17, and 4.6.8

Problem 4.2.1 Solution

The CDF of $X$ is

$$F_X(x) = \begin{cases} 
0 & x < -1, \\
(x + 1)/2 & -1 \leq x < 1, \\
1 & x \geq 1. 
\end{cases} \quad (1)$$

Each question can be answered by expressing the requested probability in terms of $F_X(x)$.

(a)

$$P [X > 1/2] = 1 - P [X \leq 1/2]$$
$$= 1 - F_X(1/2) = 1 - 3/4 = 1/4. \quad (2)$$

(b) This is a little trickier than it should be. Being careful, we can write

$$P [-1/2 \leq X < 3/4] = P [-1/2 < X \leq 3/4]$$
$$+ P [X = -1/2] - P [X = 3/4]. \quad (3)$$

Since the CDF of $X$ is a continuous function, the probability that $X$ takes on any specific value is zero. This implies $P[X = 3/4] = 0$ and $P[X = -1/2] = 0$. (If this is not clear at this point, it will become clear in Section 4.7.) Thus,

$$P [-1/2 \leq X < 3/4] = P [-1/2 < X \leq 3/4]$$
$$= F_X(3/4) - F_X(-1/2) = 5/8. \quad (4)$$
(c) \[ P[|X| \leq 1/2] = P[-1/2 \leq X \leq 1/2] \]
\[= P[X \leq 1/2] - P[X < -1/2]. \quad (5) \]
Note that \( P[X \leq 1/2] = F_X(1/2) = 3/4. \) Since the probability that \( P[X = -1/2] \)
0, \( P[X < -1/2] = P[X \leq 1/2]. \) Hence \( P[X < -1/2] = F_X(-1/2) = 1/4. \)
This implies
\[ P[|X| \leq 1/2] = P[X \leq 1/2] - P[X < -1/2] \]
\[= 3/4 - 1/4 = 1/2. \quad (6) \]

(d) Since \( F_X(1) = 1, \) we must have \( a \leq 1. \) For \( a \leq 1, \) we need to satisfy
\[ P[X \leq a] = F_X(a) = \frac{a + 1}{2} = 0.8. \quad (7) \]
Thus \( a = 0.6. \)

**Problem 4.3.1 Solution**

\[ f_X(x) = \begin{cases} \]
\[ cx & 0 \leq x \leq 2, \\
\[ 0 & \text{otherwise}. \]
\[ \end{cases} \quad (1) \]

(a) From the above PDF we can determine the value of \( c \) by integrating the PDF and setting it equal to 1, yielding
\[ \int_0^2 cx \, dx = 2c = 1. \quad (2) \]
Therefore \( c = 1/2. \)

(b) \( P[0 \leq X \leq 1] = \int_0^1 \frac{x}{2} \, dx = 1/4. \)

(c) \( P[-1/2 \leq X \leq 1/2] = \int_{-1/2}^{1/2} \frac{x}{2} \, dx = 1/16. \)

(d) The CDF of \( X \) is found by integrating the PDF from 0 to \( x. \)
\[ F_X(x) = \int_0^x f_X(x') \, dx' = \begin{cases} \]
\[ 0 & x < 0, \\
\[ x^2/4 & 0 \leq x \leq 2, \\
\[ 1 & x > 2. \]
\[ \end{cases} \quad (3) \]
Problem 4.3.4 Solution

For $x < 0$, $F_X(x) = 0$. For $x \geq 0$,
\[
F_X(x) = \int_0^x f_X(y) \, dy = \int_0^x a^2 y e^{-a^2 y^2/2} \, dy = -e^{-a^2 y^2/2}\bigg|_0^x = 1 - e^{-a^2 x^2/2}. \quad (1)
\]

A complete expression for the CDF of $X$ is
\[
F_X(x) = \begin{cases} 
0 & x < 0, \\
1 - e^{-a^2 x^2/2} & x \geq 0 
\end{cases} \quad (2)
\]

Problem 4.4.2 Solution

(a) Since the PDF is uniform over $[1,9]$
\[
E[X] = \frac{1 + 9}{2} = 5, \quad \text{Var}[X] = \frac{(9 - 1)^2}{12} = \frac{16}{3}. \quad (1)
\]

(b) Define $h(X) = 1/\sqrt{X}$ then
\[
h(E[X]) = 1/\sqrt{5}, \quad (2)
\]
\[
E[h(X)] = \int_1^9 \frac{x^{-1/2}}{8} \, dx = 1/2. \quad (3)
\]

(c)
\[
E[Y] = E[h(X)] = 1/2, \quad (4)
\]
\[
\text{Var}[Y] = E[Y^2] - (E[Y])^2 = \int_1^9 \frac{x^{-1}}{8} \, dx - E[X]^2 = \frac{\ln 9}{8} - 1/4. \quad (5)
\]
Problem 4.4.4 Solution

We can find the expected value of $X$ by direct integration of the given PDF.

\[ f_Y(y) = \begin{cases} 
  y/2 & 0 \leq y \leq 2, \\
  0 & \text{otherwise.} 
\end{cases} \quad (1) \]

The expectation is

\[ E[Y] = \int_0^2 \frac{y^2}{2} \, dy = \frac{4}{3}. \quad (2) \]

To find the variance, we first find the second moment

\[ E[Y^2] = \int_0^2 \frac{y^3}{2} \, dy = 2. \quad (3) \]

The variance is then

\[ \text{Var}[Y] = E[Y^2] - E[Y]^2 = 2 - \left( \frac{4}{3} \right)^2 = \frac{2}{9}. \]

Problem 4.4.6 Solution

To evaluate the moments of $V$, we need the PDF $f_V(v)$, which we find by taking the derivative of the CDF $F_V(v)$. The CDF and corresponding PDF of $V$ are

\[ F_V(v) = \begin{cases} 
  0 & v < -5, \\
  (v + 5)^2/144 & -5 \leq v < 7, \\
  1 & v \geq 7, 
\end{cases} \quad (1) \]

\[ f_V(v) = \begin{cases} 
  0 & v < -5, \\
  (v + 5)/72 & -5 \leq v < 7, \\
  0 & v \geq 7. 
\end{cases} \quad (2) \]

(a) The expected value of $V$ is

\[ E[V] = \int_{-\infty}^{\infty} vf_V(v) \, dv = \frac{1}{72} \int_{-5}^{7} (v^2 + 5v) \, dv 
\]

\[ = \frac{1}{72} \left( \frac{v^3}{3} + \frac{5v^2}{2} \right) \Big|_{-5}^{7} \]

\[ = \frac{1}{72} \left( \frac{343}{3} + \frac{245}{2} + \frac{125}{3} - \frac{125}{2} \right) = 3. \quad (3) \]
(b) To find the variance, we first find the second moment
\[
E[V^2] = \int_{-\infty}^{\infty} v^2 f_V(v) \, dv = \frac{1}{72} \int_{-5}^{7} (v^3 + 5v^2) \, dv
\]
\[
= \frac{1}{72} \left( \frac{v^4}{4} + \frac{5v^3}{3} \right) \bigg|_{-5}^{7}
\]
\[
= 6719/432 = 15.55. \tag{4}
\]
The variance is \( \text{Var}[V] = E[V^2] - (E[V])^2 = 2831/432 = 6.55. \)

(c) The third moment of \( V \) is
\[
E[V^3] = \int_{-\infty}^{\infty} v^3 f_V(v) \, dv
\]
\[
= \frac{1}{72} \int_{-5}^{7} (v^4 + 5v^3) \, dv
\]
\[
= \frac{1}{72} \left( \frac{v^5}{5} + \frac{5v^4}{4} \right) \bigg|_{-5}^{7} = 86.2. \tag{5}
\]

**Problem 4.5.6 Solution**

From Appendix A, an Erlang random variable \( X \) with parameters \( \lambda > 0 \) and \( n \) has PDF
\[
f_X(x) = \begin{cases} 
\lambda^n x^{n-1} e^{-\lambda x}/(n-1)! & x \geq 0, \\
0 & \text{otherwise}.
\end{cases} \tag{1}
\]

In addition, the mean and variance of \( X \) are
\[
E[X] = \frac{n}{\lambda}, \quad \text{Var}[X] = \frac{n}{\lambda^2}. \tag{2}
\]

(a) Since \( \lambda = 1/3 \) and \( E[X] = n/\lambda = 15 \), we must have \( n = 5 \).

(b) Substituting the parameters \( n = 5 \) and \( \lambda = 1/3 \) into the given PDF, we obtain
\[
f_X(x) = \begin{cases} 
(1/3)^5 x^4 e^{-x/3}/24 & x \geq 0, \\
0 & \text{otherwise}.
\end{cases} \tag{3}
\]

(c) From above, we know that \( \text{Var}[X] = n/\lambda^2 = 45. \)
(a) The PDF of a continuous uniform \((-5, 5)\) random variable is

\[
f_X(x) = \begin{cases} 
  1/10 & -5 \leq x \leq 5, \\
  0 & \text{otherwise}.
\end{cases}
\]

(1)

(b) For \(x < -5\), \(F_X(x) = 0\). For \(x \geq 5\), \(F_X(x) = 1\). For \(-5 \leq x \leq 5\), the CDF is

\[
F_X(x) = \int_{-5}^{x} f_X(\tau) \, d\tau = \frac{x + 5}{10}.
\]

(2)

The complete expression for the CDF of \(X\) is

\[
F_X(x) = \begin{cases} 
  0 & x < -5, \\
  (x + 5)/10 & -5 \leq x \leq 5, \\
  1 & x > 5.
\end{cases}
\]

(3)

(c) The expected value of \(X\) is

\[
\int_{-5}^{5} \frac{x}{10} \, dx = \frac{x^2}{20}\bigg|_{-5}^{5} = 0.
\]

(4)

Another way to obtain this answer is to use Theorem 4.6 which says the expected value of \(X\) is \(E[X] = (5 + -5)/2 = 0\).

(d) The fifth moment of \(X\) is

\[
\int_{-5}^{5} \frac{x^5}{10} \, dx = \frac{x^6}{60}\bigg|_{-5}^{5} = 0.
\]

(5)

(e) The expected value of \(e^X\) is

\[
\int_{-5}^{5} \frac{e^x}{10} \, dx = \frac{e^x}{10}\bigg|_{-5}^{5} = \frac{e^5 - e^{-5}}{10} = 14.84.
\]

(6)
Problem 4.5.17 Solution

For an Erlang \((n, \lambda)\) random variable \(X\), the \(k\)th moment is

\[
E \left[ X^k \right] = \int_0^\infty x^k f_X(x) \, dx = \int_0^\infty \frac{\lambda^n x^{n+k-1}}{(n-1)!} e^{-\lambda x} \, dx = \left( n + k - 1 \right)! \frac{\lambda^k}{\lambda^k (n-1)!} \int_0^\infty \frac{\lambda^{n+k} x^{n+k-1}}{(n+k-1)!} e^{-\lambda x} \, dx. \tag{1}
\]

The above integral equals 1 since it is the integral of an Erlang \((n+k, \lambda)\) PDF over all possible values. Hence,

\[
E \left[ X^k \right] = \frac{(n + k - 1)!}{\lambda^k (n-1)!}. \tag{2}
\]

This implies that the first and second moments are

\[
E[X] = \frac{n!}{(n-1)! \lambda} = \frac{n}{\lambda}, \quad E[X^2] = \frac{(n+1)!}{\lambda^2 (n-1)!} = \frac{(n+1)n}{\lambda^2}. \tag{3}
\]

It follows that the variance of \(X\) is \(n/\lambda^2\).

Problem 4.6.8 Solution

Repeating Definition 4.11,

\[
Q(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-u^2/2} \, du. \tag{1}
\]

Making the substitution \(x = u/\sqrt{2}\), we have

\[
Q(z) = \frac{1}{\sqrt{\pi}} \int_{z/\sqrt{2}}^\infty e^{-x^2} \, dx = \frac{1}{2} \text{erfc} \left( \frac{z}{\sqrt{2}} \right). \tag{2}
\]