Yates and Goodman 3e Solution Set: 5.1.1, 5.2.2, 5.2.3, 5.3.1, 5.3.5, 5.3.6, 5.4.2, 5.4.4, 5.5.1, 5.5.5, and 5.5.9

Problem 5.1.1 Solution

(a) The probability $P[X \leq 2, Y \leq 3]$ can be found by evaluating the joint CDF $F_{X,Y}(x,y)$ at $x=2$ and $y=3$. This yields

$$P[X \leq 2, Y \leq 3] = F_{X,Y}(2,3) = (1 - e^{-2})(1 - e^{-3}) \quad (1)$$

(b) To find the marginal CDF of $X$, $F_X(x)$, we simply evaluate the joint CDF at $y = \infty$.

$$F_X(x) = F_{X,Y}(x,\infty) = \begin{cases} 1 - e^{-x} & x \geq 0, \\ 0 & \text{otherwise}. \end{cases} \quad (2)$$

(c) Likewise for the marginal CDF of $Y$, we evaluate the joint CDF at $X = \infty$.

$$F_Y(y) = F_{X,Y}(\infty,y) = \begin{cases} 1 - e^{-y} & y \geq 0, \\ 0 & \text{otherwise}. \end{cases} \quad (3)$$

Problem 5.2.2 Solution

On the $X,Y$ plane, the joint PMF is

$$P_{X,Y}(x,y)$$

- $P_{X,Y}(2c,1) = c$
- $P_{X,Y}(3c,1) = c$
- $P_{X,Y}(2c,2c) = 3c$
- $P_{X,Y}(3c,2c) = 2c$
- $P_{X,Y}(2c,3c) = 3c$
- $P_{X,Y}(3c,3c) = c$

1
(a) To find $c$, we sum the PMF over all possible values of $X$ and $Y$. We choose $c$ so the sum equals one.

$$
\sum_{x} \sum_{y} P_{X,Y}(x,y) = \sum_{x=-2,0,2} \sum_{y=-1,0,1} c \mid x + y \mid = 6c + 2c + 6c = 14c.
$$

Thus $c = 1/14$.

(b) 

$$
P[Y < X] = P_{X,Y}(0,-1) + P_{X,Y}(2,-1) + P_{X,Y}(2,0) + P_{X,Y}(2,1)
= c + c + 2c + 3c = 7c = 1/2.
$$

(c) 

$$
P[Y > X] = P_{X,Y}(-2,-1) + P_{X,Y}(-2,0) + P_{X,Y}(-2,1) + P_{X,Y}(0,1)
= 3c + 2c + c + c = 7c = 1/2.
$$

(d) From the sketch of $P_{X,Y}(x,y)$ given above, $P[X = Y] = 0$.

(e) 

$$
P[X < 1] = P_{X,Y}(-2,-1) + P_{X,Y}(-2,0) + P_{X,Y}(-2,1)
+ P_{X,Y}(0,-1) + P_{X,Y}(0,1)
= 8c = 8/14.
$$

Problem 5.2.3 Solution

Let $r$ (reject) and $a$ (accept) denote the result of each test. There are four possible outcomes: $rr, ra, ar, aa$. The sample tree is

```
p   r  \bullet rr  p^2
p
1-p  a  \bullet ra  p(1-p)

p  r  \bullet ar  p(1-p)
1-p  a  \bullet aa  (1-p)^2

1-p  a  \bullet aa  (1-p)^2
```

Now we construct a table that maps the sample outcomes to values of $X$ and $Y$.

<table>
<thead>
<tr>
<th>outcome</th>
<th>$P[.]$</th>
<th>$X$</th>
<th>$Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$rr$</td>
<td>$p^2$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$ra$</td>
<td>$p(1-p)$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$ar$</td>
<td>$p(1-p)$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$aa$</td>
<td>$(1-p)^2$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

This table is essentially the joint PMF $P_{X,Y}(x,y)$.

\[
P_{X,Y}(x,y) = \begin{cases} 
  p^2 & x = 1, y = 1, \\
  p(1-p) & x = 0, y = 1, \\
  p(1-p) & x = 1, y = 0, \\
  (1-p)^2 & x = 0, y = 0, \\
  0 & \text{otherwise.}
\end{cases}
\]  

**Problem 5.3.1 Solution**

On the $X,Y$ plane, the joint PMF $P_{X,Y}(x,y)$ is

![Joint PMF](image)

By choosing $c = 1/28$, the PMF sums to one.
(a) The marginal PMFs of X and Y are

\[ P_X(x) = \sum_{y=1,3} P_{X,Y}(x,y) = \begin{cases} 
4/28 & x = 1, \\
8/28 & x = 2, \\
16/28 & x = 4, \\
0 & \text{otherwise}, 
\end{cases} \]  \hfill (1)

\[ P_Y(y) = \sum_{x=1,2,4} P_{X,Y}(x,y) = \begin{cases} 
7/28 & y = 1, \\
21/28 & y = 3, \\
0 & \text{otherwise}. 
\end{cases} \]  \hfill (2)

(b) The expected values of X and Y are

\[ E[X] = \sum_{x=1,2,4} xP_X(x) = (4/28) + 2(8/28) + 4(16/28) = 3, \]  \hfill (3)

\[ E[Y] = \sum_{y=1,3} yP_Y(y) = 7/28 + 3(21/28) = 5/2. \]  \hfill (4)

(c) The second moments are

\[ E[X^2] = \sum_{x=1,2,4} xP_X(x) \\
= 1^2(4/28) + 2^2(8/28) + 4^2(16/28) = 73/7, \]  \hfill (5)

\[ E[Y^2] = \sum_{y=1,3} yP_Y(y) = 1^2(7/28) + 3^2(21/28) = 7. \]  \hfill (6)

The variances are

\[ \text{Var}[X] = E[X^2] - (E[X])^2 = 10/7, \]  \hfill (7)

\[ \text{Var}[Y] = E[Y^2] - (E[Y])^2 = 3/4. \]  \hfill (8)

The standard deviations are \( \sigma_X = \sqrt{10/7} \) and \( \sigma_Y = \sqrt{3/4} \).
Problem 5.3.5 Solution

The joint PMF of $N, K$ is

$$P_{N,K}(n,k) = \begin{cases} (1 - p)^{n-1} p/n & k = 1, 2, \ldots, n, \\ n = 1, 2 \ldots, & \\ o & \text{otherwise.} \end{cases} \quad (1)$$

For $n \geq 1$, the marginal PMF of $N$ is

$$P_N(n) = \sum_{k=1}^{n} P_{N,K}(n,k) = \sum_{k=1}^{n} (1 - p)^{n-1} p/n = (1 - p)^{n-1} p. \quad (2)$$

The marginal PMF of $K$ is found by summing $P_{N,K}(n,k)$ over all possible $N$. Note that if $K = k$, then $N \geq k$. Thus,

$$P_K(k) = \sum_{n=k}^{\infty} \frac{1}{n} (1 - p)^{n-1} p. \quad (3)$$

Unfortunately, this sum cannot be simplified.

Problem 5.3.6 Solution

For $n = 0, 1, \ldots$, the marginal PMF of $N$ is

$$P_N(n) = \sum_{k} P_{N,K}(n,k) = \sum_{k=0}^{n} \frac{100^n e^{-100}}{(n+1)!} = \frac{100^n e^{-100}}{n!}. \quad (1)$$

For $k = 0, 1, \ldots$, the marginal PMF of $K$ is

$$P_K(k) = \sum_{n=k}^{\infty} \frac{100^n e^{-100}}{(n+1)!} = \frac{1}{100} \sum_{n=k}^{\infty} \frac{100^{n+1} e^{-100}}{(n+1)!}$$

$$= \frac{1}{100} \sum_{n=k}^{\infty} P_N(n + 1)$$

$$= P [N > k] / 100. \quad (2)$$
Problem 5.4.2 Solution

We are given the joint PDF

\[ f_{X,Y}(x,y) = \begin{cases} 
  cxy^2 & 0 \leq x, y \leq 1, \\
  0 & \text{otherwise.}
\end{cases} \tag{1} \]

(a) To find the constant \( c \) integrate \( f_{X,Y}(x,y) \) over the all possible values of \( X \) and \( Y \) to get

\[ 1 = \int_0^1 \int_0^1 cxy^2 \, dx \, dy = c/6. \tag{2} \]

Therefore \( c = 6 \).

(b) The probability \( P[X \geq Y] \) is the integral of the joint PDF \( f_{X,Y}(x,y) \) over the indicated shaded region.

\[ P[X \geq Y] = \int_0^1 \int_0^x 6xy^2 \, dy \, dx = \int_0^1 2x^4 \, dx = 2/5. \tag{3} \]

Similarly, to find \( P[Y \leq X^2] \) we can integrate over the region shown in the figure.

\[ P[Y \leq X^2] = \int_0^1 \int_0^{x^2} 6xy^2 \, dy \, dx = 1/4. \tag{4} \]
(c) Here we can choose to either integrate \( f_{X,Y}(x, y) \) over the lighter shaded region, which would require the evaluation of two integrals, or we can perform one integral over the darker region by recognizing

\[
P[\min(X, Y) \leq 1/2] = 1 - P[\min(X, Y) > 1/2]
\]

\[
= 1 - \int_{1/2}^{1} \int_{1/2}^{1} 6xy^2 \, dx \, dy
\]

\[
= 1 - \int_{1/2}^{1} 9y^2 \frac{1}{4} \, dy
\]

\[
= \frac{11}{32}.
\]  

(d) The probability \( P[\max(X, Y) \leq 3/4] \) can be found by integrating over the shaded region shown below.

\[
P[\max(X, Y) \leq 3/4] = P[X \leq 3/4, Y \leq 3/4]
\]

\[
= \int_{0}^{3/4} \int_{0}^{3/4} 6xy^2 \, dx \, dy
\]

\[
= \left( \frac{x^2}{0} \right) \left( \frac{y^3}{0} \right)
\]

\[
= (3/4)^5 = 0.237.
\]  

**Problem 5.4.4 Solution**

The only difference between this problem and Example 5.8 is that in this problem we must integrate the joint PDF over the regions to find the probabilities. Just as in Example 5.8, there are five cases. We will use variable \( u \) and \( v \) as dummy variables for \( x \) and \( y \).

- \( x < 0 \) or \( y < 0 \)
In this case, the region of integration doesn’t overlap the region of nonzero probability and

\[ F_{X,Y}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u, v) \, du \, dv = 0. \]  

\[ (1) \]

- **0 < y ≤ x ≤ 1**

In this case, the region where the integral has a nonzero contribution is

\[ F_{X,Y}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u, v) \, dy \, dx \]
\[ = \int_{0}^{y} \int_{v}^{x} 8uv \, du \, dv \]
\[ = \int_{0}^{y} 4(x^2 - v^2) \, v \, dv \]
\[ = 2x^2v^2 - v^4 \bigg|_{v=0}^{v=y} \]
\[ = 2x^2y^2 - y^4. \]  

\[ (2) \]

- **0 < x ≤ y and 0 ≤ x ≤ 1**
\[ F_{X,Y}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u, v) \, dv \, du \]
\[ = \int_{0}^{y} \int_{0}^{u} 8uv \, dv \, du \]
\[ = \int_{0}^{y} 4u^3 \, du = x^4. \quad (3) \]

- \(0 < y \leq 1\) and \(x \geq 1\)

\[ F_{X,Y}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u, v) \, dv \, du \]
\[ = \int_{0}^{y} \int_{0}^{1} 8uv \, dv \, du \]
\[ = \int_{0}^{y} 4v(1 - v^2) \, dv \]
\[ = 2y^2 - y^4. \quad (4) \]

- \(x \geq 1\) and \(y \geq 1\)
In this case, the region of integration completely covers the region of nonzero probability and

\[ F_{X,Y}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u, v) \, du \, dv = 1. \]  

(5)

The complete answer for the joint CDF is

\[ F_{X,Y}(x, y) = \begin{cases} 
0 & x < 0 \text{ or } y < 0, \\
2x^2y^2 - y^4 & 0 < y \leq x \leq 1, \\
x^4 & 0 \leq x \leq y, 0 \leq x \leq 1, \\
2y^2 - y^4 & 0 \leq y \leq 1, x \geq 1, \\
1 & x \geq 1, y \geq 1.
\]  

(6)

Problem 5.5.1 Solution

The joint PDF (and the corresponding region of nonzero probability) are

\[ f_{X,Y}(x, y) = \begin{cases} 
1/2 & -1 \leq x \leq y \leq 1, \\
0 & \text{otherwise.}
\]  

(1)

(a)

\[ P[X > 0] = \int_{0}^{1} \int_{x}^{1} \frac{1}{2} dy \, dx = \int_{0}^{1} \frac{1 - x}{2} \, dx = 1/4 \]  

(2)

This result can be deduced by geometry. The shaded triangle of the \( X,Y \) plane corresponding to the event \( X > 0 \) is 1/4 of the total shaded area.
(b) For $x > 1$ or $x < -1$, $f_X(x) = 0$. For $-1 \leq x \leq 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dy = \int_x^1 \frac{1}{2} \, dy = (1 - x)/2. \quad (3)$$

The complete expression for the marginal PDF is

$$f_X(x) = \begin{cases} (1 - x)/2 & -1 \leq x \leq 1, \\ 0 & \text{otherwise}. \end{cases} \quad (4)$$

(c) From the marginal PDF $f_X(x)$, the expected value of $X$ is

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) \, dx = \frac{1}{2} \int_{-1}^{1} x(1 - x) \, dx$$

$$= \frac{x^2}{4} - \frac{x^3}{6} \bigg|_{-1}^{1} = -\frac{1}{3}. \quad (5)$$

Problem 5.5.5 Solution

The joint PDF of $X$ and $Y$ and the region of nonzero probability are

![Joint PDF Region](image)

$$f_{X,Y}(x,y) = \begin{cases} \frac{5x^2}{2} & -1 \leq x \leq 1, 0 \leq y \leq x^2, \\ 0 & \text{otherwise}. \end{cases} \quad (1)$$

We can find the appropriate marginal PDFs by integrating the joint PDF.

(a) The marginal PDF of $X$ is

$$f_X(x) = \int_0^{x^2} \frac{5x^2}{2} \, dy = \begin{cases} \frac{5x^4}{2} & -1 \leq x \leq 1, \\ 0 & \text{otherwise}. \end{cases} \quad (2)$$
(b) Note that $f_Y(y) = 0$ for $y > 1$ or $y < 0$. For $0 \leq y \leq 1,$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{5x^2}{2} \, dx + \int_{\sqrt{y}}^{1} \frac{5x^2}{2} \, dx = 5(1 - y^{3/2})/3.$$  \hfill (3)

The complete expression for the marginal CDF of $Y$ is

$$f_Y(y) = \begin{cases} 
5(1 - y^{3/2})/3 & 0 \leq y \leq 1, \\
0 & \text{otherwise}.
\end{cases} \hfill (4)$$

**Problem 5.5.9 Solution**

(a) The joint PDF of $X$ and $Y$ and the region of nonzero probability are

$$f_{X,Y}(x, y) = \begin{cases} 
cy & 0 \leq y \leq x \leq 1, \\
0 & \text{otherwise}.
\end{cases} \hfill (1)$$

(b) To find the value of the constant, $c$, we integrate the joint PDF over all $x$ and $y$.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx \, dy = \int_{0}^{1} \int_{0}^{x} cy \, dy \, dx = \int_{0}^{1} \frac{cx^2}{2} \, dx$$

$$= \frac{cx^3}{6} \bigg|_{0}^{1} = \frac{c}{6}. \hfill (2)$$

Thus $c = 6.$
(c) We can find the CDF $F_X(x) = P[X \leq x]$ by integrating the joint PDF over the event $X \leq x$. For $x < 0$, $F_X(x) = 0$. For $x > 1$, $F_X(x) = 1$. For $0 \leq x \leq 1$,

$$F_X(x) = \int \int_{x' \leq x} f_{X,Y}(x',y') \, dy' \, dx'$$
$$= \int_0^x \int_0^{x'} 6y' \, dy' \, dx'$$
$$= \int_0^x 3(x')^2 \, dx' = x^3. \quad (3)$$

The complete expression for the joint CDF is

$$F_X(x) = \begin{cases} 0 & x < 0, \\ x^3 & 0 \leq x \leq 1, \\ 1 & x \geq 1. \end{cases} \quad (4)$$

(d) Similarly, we find the CDF of $Y$ by integrating $f_{X,Y}(x,y)$ over the event $Y \leq y$. For $y < 0$, $F_Y(y) = 0$ and for $y > 1$, $F_Y(y) = 1$. For $0 \leq y \leq 1$,

$$F_Y(y) = \int \int_{y' \leq y} f_{X,Y}(x',y') \, dy' \, dx'$$
$$= \int_0^y \int_0^{1-y'} 6y' \, dx' \, dy'$$
$$= \int_0^y 6y'(1-y') \, dy'$$
$$= 3(y')^2 - 2(y')^3 \bigg|_0^y = 3y^2 - 2y^3. \quad (5)$$

The complete expression for the CDF of $Y$ is

$$F_Y(y) = \begin{cases} 0 & y < 0, \\ 3y^2 - 2y^3 & 0 \leq y \leq 1, \\ 1 & y > 1. \end{cases} \quad (6)$$
(e) To find $P[Y \leq X/2]$, we integrate the joint PDF $f_{X,Y}(x,y)$ over the region $y \leq x/2$.

$$P[Y \leq X/2] = \int_0^1 \int_0^{x/2} 6y \, dy \, dx$$

$$= \int_0^1 3y^2 \bigg|_0^{x/2} \, dx$$

$$= \int_0^1 \frac{3x^2}{4} \, dx = \frac{1}{4}.$$  \hfill (7)