PROBABILITY AND STOCHASTIC PROCESSES
A FRIENDLY INTRODUCTION FOR ELECTRICAL AND COMPUTER ENGINEERS
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Chapter 9 & 10 Viewgraphs

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Sums of RVs

- $W_n = X_1 + \cdots + X_n$

- Could use $f_{X_1, \ldots, X_N}(x_1, \ldots, x_n)$

- Special techniques
  - For $E[W]$ and $\text{Var}[W]$
  - $X_1, \ldots, X_n$ i.i.d
  - Limit theorems for large $n$
Expectations of Sums

- **Theorem:** For any $X_1, \ldots, X_n$,

\[ E[W_n] = E[X_1] + E[X_2] + \cdots + E[X_n] \]

- Proof has been done in class
Variance of the Sum

- $W = X_1 + \cdots + X_n$ has variance

$$\text{Var}[W_n] = \sum_{i=1}^{n} \text{Var}[X_i] + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \text{Cov}[X_i, X_j]$$

(Proof by Algebra)

- $X_1, \ldots, X_n$ are independent,

$$\text{Var}[W_n] = \text{Var}[X_1] + \cdots + \text{Var}[X_n]$$
PDF of the Sum

PDF of $W = X + Y$ is

$$f_W(w) = \int_{-\infty}^{\infty} f_{X,Y}(x, w-x) \, dx$$

$$= \int_{-\infty}^{\infty} f_{X,Y}(w-y, y) \, dy$$
PDF of an Independent Sum

- $X$ and $Y$ are independent,

\[
    f_W(w) = \int_{-\infty}^{\infty} f_X(w - y) f_Y(y) \, dy
\]

\[
    = \int_{-\infty}^{\infty} f_X(x) f_Y(w - x) \, dx
\]

- PDF of an independent sum is the convolution of the PDFs
Moment Generating Function (MGF)

- The moment generating function (MGF) of $X$ is
  \[ \phi_X(s) = E[e^{sX}] \]

- $X$ continuous: \[ \phi_X(s) = \int_{-\infty}^{\infty} e^{sx} f_X(x) \, dx \]

- $Y$ discrete: \[ \phi_Y(s) = \sum_{y_i \in S_Y} e^{s y_i} P_Y(y_i) \]

- PDF $\leftrightarrow$ MGF
MGF Properties

- $\phi_X(s)\big|_{s=0} = 1$
- $Y = aX + b$ has MGF $\phi_Y(s) = e^{sb}\phi_X(as)$
- $X$ has $n$th moment

$$E[X^n] = \frac{d^n\phi_X(s)}{ds^n}\bigg|_{s=0}$$

- **Theorem:** $X_1, \ldots, X_n$ i.i.d RVs $\implies$ MGF of $W_n = X_1 + \cdots + X_n$ is

$$\phi_{W_n}(s) \xhookrightarrow{\text{i.}} \prod_{i=1}^n \phi_{X_i}(s) \xhookrightarrow{\text{i.d}} (\phi_X(s))^n$$
MGF Examples

• If $X = a$, then $f_X(x) = \delta(x - a)$, and

$$\phi_X(s) = \int_{-\infty}^{\infty} e^{sx} \delta(x - a) \, dx = e^{sa}$$

• $X$ with uniform PDF $f_X(x) = \begin{cases} 
1 & 0 \leq x \leq 1 \\
0 & \text{otherwise}
\end{cases}$ has MGF

$$\phi_X(s) = \int_{0}^{1} e^{sx} \, dx = \frac{e^{s} - 1}{s}$$
Exponential MGF

- Let $X$ have the exponential PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The MGF of $X$ is

$$\phi_X(s) = \int_0^\infty e^{sx} \lambda e^{-\lambda x} \, dx = \frac{\lambda}{\lambda - s}$$
Bernoulli MGF

- $X$ is Bernoulli:

\[
P_X(x) = \begin{cases} 
1 - p & x = 0 \\
p & x = 1 \\
0 & \text{otherwise}
\end{cases}
\]

The MGF of $X$ is

\[
\phi_X(s) = E[e^{sX}] = (1 - p)e^0 + pe^s = 1 - p + pe^s
\]
Geometric PMF

- $N$ is geometric:

$$P_N(n) = \begin{cases} 
(1 - p)^{n-1}p & n = 1, 2, \ldots \\
0 & \text{otherwise}
\end{cases}$$

- The MGF of $N$ is

$$\phi_N(s) = \sum_{n=1}^{\infty} e^{sn}p(1 - p)^{n-1}$$

$$= pe^s \sum_{n=1}^{\infty} ((1 - p)e^s)^{n-1}$$

$$= \frac{pe^s}{1 - (1 - p)e^s}$$
Poisson PMF

- $K$ is Poisson:

$$P_K(k) = \begin{cases} \frac{\alpha^k e^{-\alpha}}{k!} & k = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases}$$

- The MGF of $K$ is

$$\phi_K(s) = \sum_{k=0}^{\infty} e^{sk} \frac{\alpha^k e^{-\alpha}}{k!}$$

$$= e^{-\alpha} \sum_{k=0}^{\infty} \left(\alpha e^s\right)^k / k!$$

$$= e^{\alpha(e^s-1)}$$
Poisson sum

- $K_1, \ldots, K_n$ independent Poisson RVs, $E[K_i] = \alpha_i$.
- MGF of $W = K_1 + \cdots + K_n$?
- $K_i$ has MGF $\phi_{K_i}(s) = e^{\alpha_i(e^s-1)} \iff$ we have just shown
  \[
  \phi_W(s) = e^{\alpha_1(e^s-1)} e^{\alpha_2(e^s-1)} \cdots e^{\alpha_n(e^s-1)}
  = e^{(\alpha_1+\cdots+\alpha_n)(e^s-1)}
  = e^{(\alpha_T)(e^s-1)}
  \]
- $W$ is Poisson with mean $\alpha_T = \alpha_1 + \cdots + \alpha_n$
**Gaussian MGF**

- **Theorem:** If $Z$ is $N[0, 1]$, then
  \[
  \phi_Z(s) = e^{s^2/2}
  \]

- **Proof:** MGF of $Z$ is
  \[
  \phi_Z(s) = \int_{-\infty}^{\infty} e^{sz} f_Z(z) \, dz
  \]
  \[
  = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{sz} e^{-z^2/2} \, dz
  \]

- **Completing the square:**
  \[
  \phi_Z(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} (z^2 - 2sz + s^2)} e^{s^2/2} \, dz
  \]
  \[
  = e^{s^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} (z-s)^2} \, dz
  \]
  \[
  = e^{s^2/2}
  \]
More on Gaussian MGF

- If $X$ is $N[\mu, \sigma^2]$ $\implies X = \sigma Z + \mu$,

- Therefore

$$\phi_X(s) = e^{s\mu} \phi_Z(\sigma s) = e^{s\mu + \sigma^2 s^2 / 2}$$
Sum of i.i.d Gaussian RVs

- Theorem: $W_n = X_1 + \cdots + X_n$ is Gaussian with and variance

\[
E[W_n] = E[X_1] + \cdots + E[X_n] \\
\text{Var}[W_n] = \text{Var}[X_1] + \cdots + \text{Var}[X_n]
\]
Random Sum of RVs

- Sum of i.i.d random variables
  \[ R = X_1 + \cdots + X_N \]

- \( N \) is random!

- MGF will be simple when \( N \) is independent of \( X_1, X_2, \ldots \)
Examples: Random Sum

- Example 1:
  \[ K_i = \text{no. of people on bus } i \]
  \[ N = \text{no. of buses arriving in 1 hour} \]
  \[ R = X_1 + \cdots + X_N; \text{ no. of people in 1 hour} \]

- Example 2:
  \[ N = \text{no. of data packets in one minute} \]
  Each packet is OK with prob \( p \)
  No. of OK packets is
  \[ R = X_1 + X_2 + \cdots + X_N \]

\( R \) is not binomial!
Theorem for Random Sum

- The random sum of i.i.d random variables $R_N = X_1 + X_2 + \cdots + X_N$ has moment generating function

$$\phi_{R_N}(s) = \phi_N \left( \ln \left[ \phi_X(s) \right] \right)$$

where $N$ is independent of $X_1, \ldots, X_n$
Central Limit Theorem

- **Review:** $X_1, X_2, \ldots$ i.i.d Gaussian RVs

$$W_n = X_1 + \cdots + X_n$$

is Gaussian with

$$E[W_n] = n\mu_X$$

$$\text{Var}[W_n] = n\sigma_X^2$$

- **Question:** If $X_1, X_2, \ldots$ are not Gaussian, then what is the PDF (or PMF) of $W_n$?
Sum Of Bernoulli RVs

- 50 flips of a fair coin: $X_i = 1$ is $H$ on flip $i$.

- $W_n$ is binomial

\[ P_{W_n}(w) = \begin{cases} \binom{50}{w}(1/2)^{50} & w = 0, 1, \ldots, 50 \\ 0 & \text{otherwise} \end{cases} \]

- What does this look like?
Sum of Bernoulli RVs

- **Note:** Fig. 1: Empirical and Fig. 2: Analytical
Central Limit Theorem

- Standardized rv $Z_n$:
  \[ Z_n = \frac{W_n - E[W]}{\sigma_W} = \frac{\sum_{i=1}^{n} X_i - n\mu_X}{\sqrt{n\sigma_X^2}} \]

- $E[Z_n] = 0$  \hspace{1em} $\text{Var}\ [Z_n] = 1$

- Central Limit Theorem:
  \[ \lim_{n \to \infty} F_{Z_n}(z) = \Phi(z) \]

- Usual Proof: Show MGF of $Z_n$ converges to Gaussian MGF
Applying the CLT

- For $W_n = X_1 + \cdots + X_n$,

$$F_{W_n}(w) = P\left[ \sqrt{n} \frac{\sigma^2_X}{Z_n} + n\mu_X \leq w \right] = F_{Z_n}\left( \frac{w - n\mu_X}{\sqrt{n}\sigma^2_X} \right)$$

- For large $n$, CLT says $F_{Z_n}(z) \approx \Phi(z)$.

- CLT Approximation:

$$F_{W_n}(w) \approx \Phi\left( \frac{w - n\mu_X}{\sqrt{n}\sigma^2_X} \right)$$
CLT for Uniform RVs

(a) $n = 1$

(b) $n = 2$

(c) $n = 3$

(d) $n = 4$
CLT for Binomial RVs

\[ n = 4, \ p = \frac{1}{2} \quad \quad \quad \quad \quad n = 8, \ p = \frac{1}{2} \]

\[ n = 16, \ p = \frac{1}{2} \quad \quad \quad \quad \quad n = 32, \ p = \frac{1}{2} \]
Problem 9.5.2

Internet packets can be classified as video (V) or as generic data (D). Based on a lot of observations taken by the Internet service provider, we have the following probability model: \( P[V] = 3/4, \ P[D] = 1/4. \) Data packets and video packets occur independently of one another. The random variable \( K_n \) is the number of video packets in a collection of \( n \) packets.

(a) What is \( E[K_{100}] \), the expected number of video packets in a set of 100 packets?

(b) What is \( \sigma_{K_{100}} \)?

(c) Use the central limit theorem to estimate \( P[K_{100} \geq 18] \).

(d) Use the central limit theorem to estimate \( P[16 \leq K_{100} \leq 24] \).
Problem 9.5.2 Solution

Knowing that the probability that voice call occurs is 0.8 and the probability that a data call occurs is 0.2 we can define the random variable $D_i$ as the number of data calls in a single telephone call. It is obvious that for any $i$ there are only two possible values for $D_i$, namely 0 and 1. Furthermore for all $i$ the $D_i$’s are independent and identically distributed with the following PMF.

$$P_D(d) = \begin{cases} 
0.8 & d = 0, \\
0.2 & d = 1, \\
0 & \text{otherwise.}
\end{cases}$$  \hspace{1cm} (1)

From the above we can determine that

$$E[D] = 0.2, \quad \text{Var}[D] = 0.2 - 0.04 = 0.16.$$  \hspace{1cm} (2)

[Continued]
Problem 9.5.2 Solution

With these facts, we can answer the questions posed by the problem.

(a) \( \mathbb{E}[K_{100}] = 100 \mathbb{E}[D] = 20 \).

(b) \( \text{Var}[K_{100}] = \sqrt{100 \text{Var}[D]} = \sqrt{16} = 4 \).

(c)

\[
\mathbb{P}[K_{100} \geq 18] = 1 - \Phi \left( \frac{18 - 20}{4} \right) \\
= 1 - \Phi(-1/2) = \Phi(1/2) = 0.6915.
\]  

(d)

\[
\mathbb{P}[16 \leq K_{100} \leq 24] = \Phi \left( \frac{24 - 20}{4} \right) - \Phi \left( \frac{16 - 20}{4} \right) \\
= \Phi(1) - \Phi(-1) \\
= 2\Phi(1) - 1 = 0.6826.
\]
Problem 9.5.3

The duration of a cellular telephone call is an exponential random variable with expected value 150 seconds. A subscriber has a calling plan that includes 300 minutes per month at a cost of $30.00 plus $0.40 for each minute that the total calling time exceeds 300 minutes. In a certain month, the subscriber has 120 cellular calls.

(a) Use the central limit theorem to estimate the probability that the subscriber’s bill is greater than $36. (Assume that the durations of all phone calls are mutually independent and that the telephone company measures call duration exactly and charges accordingly, without rounding up fractional minutes.)

(b) Suppose the telephone company does charge a full minute for each fractional minute used. Re-calculate your estimate of the probability that the bill is greater than $36.
Problem 9.5.3 Solution

(a) Let $X_1, \ldots, X_{120}$ denote the set of call durations (measured in minutes) during the month. From the problem statement, each $X - I$ is an exponential ($\lambda$) random variable with $E[X_i] = 1/\lambda = 2.5$ min and $\text{Var}[X_i] = 1/\lambda^2 = 6.25$ min$^2$. The total number of minutes used during the month is $Y = X_1 + \cdots + X_{120}$. By Theorem 9.1 and Theorem 9.3,

$$E[Y] = 120 E[X_i] = 300$$
$$\text{Var}[Y] = 120 \text{Var}[X_i] = 750.$$

The subscriber's bill is $30 + 0.4(y - 300)^+$ where $x^+ = x$ if $x \geq 0$ or $x^+ = 0$ if $x < 0$. The subscribers bill is exactly $36$ if $Y = 315$. The probability the subscribers bill exceeds $36$ equals

$$P[Y > 315] = P\left[\frac{Y - 300}{\sigma_Y} > \frac{315 - 300}{\sigma_Y}\right]$$
$$= Q\left(\frac{15}{\sqrt{750}}\right) = 0.2919. \quad (2)$$

[Continued]
(b) If the actual call duration is $X_i$, the subscriber is billed for $M_i = \lfloor X_i \rfloor$ minutes. Because each $X_i$ is an exponential ($\lambda$) random variable, Theorem 4.9 says that $M_i$ is a geometric ($p$) random variable with $p = 1 - e^{-\lambda} = 0.3297$. Since $M_i$ is geometric,

$$E[M_i] = \frac{1}{p} = 3.033,$$

$$\text{Var}(M_i) = \frac{1-p}{p^2} = 6.167.$$  \hspace{1cm} (3)

The number of billed minutes in the month is $B = M_1 + \cdots + M_{120}$. Since $M_1, \ldots, M_{120}$ are iid random variables,

$$E[B] = 120 E[M_i] = 364.0,$$

$$\text{Var}(B) = 120 \text{Var}[M_i] = 740.08.$$  \hspace{1cm} (4)

Similar to part (a), the subscriber is billed $36 if $B = 315$ minutes. The probability the subscriber is billed more than $36 is

$$P[B > 315] = P\left[ \frac{B - 364}{\sqrt{740.08}} > \frac{315 - 365}{\sqrt{740.08}} \right]$$

$$= Q(-1.8) = \Phi(1.8) = 0.964.$$  \hspace{1cm} (5)
Theorem 10.3  Chebyshev Inequality

For an arbitrary random variable $Y$ and constant $c > 0$,

$$P [|Y - \mu_Y| \geq c] \leq \frac{\text{Var}[Y]}{c^2}.$$
**Proof: Theorem 10.3**

In the Markov inequality, Theorem 10.2, let $X = (Y - \mu_Y)^2$. The inequality states

$$
P \left[ X \geq c^2 \right] = P \left[ (Y - \mu_Y)^2 \geq c^2 \right] \leq \frac{E[(Y - \mu_Y)^2]}{c^2} = \frac{\text{Var}[Y]}{c^2}. \quad (1)$$

The theorem follows from the fact that $\{(Y - \mu_Y)^2 \geq c^2\} = \{|Y - \mu_Y| \geq c\}$. 
Section 10.5

Confidence Intervals
Confidence Intervals

\[ P \left[ | M_n(X) - \mu_X | < c \right] \geq 1 - \frac{\text{Var}[X]}{nc^2} = 1 - \alpha \]  

(1)

Equation (10.35) contains two inequalities.

- One inequality,
  \[ | M_n(X) - \mu_X | < c, \]
  defines an event.

- This event states that the sample mean is within \( \pm c \) units of the expected value.

- The length of the interval that defines this event, \( 2c \) units, is referred to as a confidence interval.

- The other inequality states that the probability that the sample mean is in the confidence interval is at least \( 1 - \alpha \).

- We refer to the quantity \( 1 - \alpha \) as the confidence coefficient.

- If \( \alpha \) is small, we are highly confident that \( M_n(X) \) is in the interval \( (\mu_X - c, \mu_X + c) \).
Problem 10.5.1

$X_1, \ldots, X_n$ are $n$ independent identically distributed samples of random variable $X$ with PMF

$$P_X(x) = \begin{cases} 
0.1 & x = 0, \\
0.9 & x = 1, \\
0 & \text{otherwise}.
\end{cases}$$

(a) How is $E[X]$ related to $P_X(1)$?

(b) Use Chebyshev's inequality to find the confidence level $\alpha$ such that $M_{90}(X)$, the estimate based on 90 observations, is within 0.05 of $P_X(1)$. In other words, find $\alpha$ such that

$$P[|M_{90}(X) - P_X(1)| \geq 0.05] \leq \alpha.$$ 

(c) Use Chebyshev's inequality to find out how many samples $n$ are necessary to have $M_n(X)$ within 0.03 of $P_X(1)$ with confidence level 0.1. In other words, find $n$ such that

$$P[|M_n(X) - P_X(1)| \geq 0.03] \leq 0.1.$$
Problem 10.5.1 Solution

\( X \) has the Bernoulli (0.9) PMF

\[
P_X(x) = \begin{cases} 
0.1 & x = 0, \\
0.9 & x = 1, \\
0 & \text{otherwise.}
\end{cases}
\] (1)

(a) \( E[X] \) is in fact the same as \( P_X(1) \) because \( X \) is Bernoulli.

(b) We can use the Chebyshev inequality to find

\[
P[|M_{90}(X) - P_X(1)| \geq 0.05] = P[|M_{90}(X) - E[X]| \geq 0.05] \leq \alpha.
\] (2)

In particular, the Chebyshev inequality states that

\[
\alpha = \frac{\sigma_X^2}{90(0.05)^2} = \frac{0.09}{90(0.05)^2} = 0.4.
\] (3)

(c) Now we wish to find the value of \( n \) such that

\[
P[|M_n(X) - P_X(1)| \geq 0.03] \leq 0.1.
\] (4)

From the Chebyshev inequality, we write

\[
0.1 = \frac{\sigma_X^2}{n(0.03)^2}.
\] (5)

Since \( \sigma_X^2 = 0.09 \), solving for \( n \) yields \( n = 100 \).
**Theorem 12.3**

Random variables $X$ and $Y$ have expected values $\mu_X$ and $\mu_Y$, standard deviations $\sigma_X$ and $\sigma_Y$, and correlation coefficient $\rho_{X,Y}$. The optimum linear mean square error (LMSE) estimator of $X$ given $Y$ is

$$
\hat{X}_L(Y) = \rho_{X,Y} \frac{\sigma_X}{\sigma_Y} (Y - \mu_Y) + \mu_X.
$$

This linear estimator has the following properties:

(a) The minimum mean square estimation error for a linear estimate is

$$
e^*_L = E \left[ (X - \hat{X}_L(Y))^2 \right] = \sigma_X^2 (1 - \rho_{X,Y}^2).$$

(b) The estimation error $X - \hat{X}_L(Y)$ is uncorrelated with $Y$. 
Proof: Theorem 12.3

Replacing \( \hat{X}_L(Y) \) by \( aY + b \) and expanding the square, we have

\[
\]  \hspace{1cm} (1)

The values of \( a \) and \( b \) that produce the minimum \( e_L \) are found by computing the partial derivatives of \( e_L \) with respect to \( a \) and \( b \) and setting the derivatives to zero, yielding

\[
\frac{\partial e_L}{\partial a} = -2 E[XY] + 2a E[Y^2] + 2b E[Y] = 0,
\]  \hspace{1cm} (2)

\[
\frac{\partial e_L}{\partial b} = -2 E[X] + 2a E[Y] + 2b = 0.
\]  \hspace{1cm} (3)

Solving the two equations for \( a \) and \( b \), we find

\[
a^* = \frac{\text{Cov}[X,Y]}{\text{Var}[Y]} = \frac{\rho_{X,Y} \sigma_X}{\sigma_Y}, \quad b^* = E[X] - a^* E[Y].
\]  \hspace{1cm} (4)

Some algebra will verify that \( a^*Y + b^* \) is the optimum linear estimate \( \hat{X}_L(Y) \). We confirm Theorem 12.3(a) by using \( \hat{X}_L(Y) \) in Equation (12.14). To prove part (b) of the theorem, observe that the correlation of \( Y \) and the estimation error is

\[
E[YX - \hat{X}_L(Y)] = E[XY] - E[Y E[X]] - \frac{\text{Cov}[X,Y]}{\text{Var}[Y]} (E[Y^2] - E[Y E[Y]])
\]

\[
= \text{Cov}[X,Y] - \frac{\text{Cov}[X,Y]}{\text{Var}[Y]} \text{Var}[Y] = 0.
\]  \hspace{1cm} (5)
Each graph contains 50 sample values of the random variable pair \((X, Y)\), each marked by the symbol \(\times\). In each graph, \(E[X] = E[Y] = 0\), \(\text{Var}[X] = \text{Var}[Y] = 1\). The solid line is the optimal linear estimator \(\hat{X}_L(Y) = \rho_{X,Y}Y\).
Problem 12.2.2

A telemetry voltage \( V \), transmitted from a position sensor on a ship’s rudder, is a random variable with PDF

\[
f_V(v) = \begin{cases} 
1/12 & \text{if } -6 \leq v \leq 6, \\
0 & \text{otherwise}. 
\end{cases}
\]

A receiver in the ship’s control room receives \( R = V + X \), The random variable \( X \) is a Gaussian \( (0, \sqrt{3}) \) noise voltage that is independent of \( V \). The receiver uses \( R \) to calculate a linear estimate of the telemetry voltage: \( \hat{V} = aR + b \). Find

(a) the expected received voltage \( \mathbb{E}[R] \),
(b) the variance \( \text{Var}[R] \) of the received voltage,
(c) the covariance \( \text{Cov}[V, R] \) of the transmitted and received voltages,
(d) \( a^* \) and \( b^* \), the optimum coefficients in the linear estimate,
(e) \( \epsilon_L^* \), the minimum mean square error of the estimate.
Problem 12.2.2 Solution

The problem statement tells us that

\[ f_{V}(v) = \begin{cases} 
    1/12 & -6 \leq v \leq 6, \\
    0 & \text{otherwise.}
\end{cases} \]  \hspace{1cm} (1)

Furthermore, we are also told that \( R = V + X \) where \( X \) is a Gaussian \((0, \sqrt{3})\) random variable.

(a) The expected value of \( R \) is the expected value \( V \) plus the expected value of \( X \). We already know that \( X \) has zero expected value, and that \( V \) is uniformly distributed between -6 and 6 volts and therefore also has zero expected value. So

\[ E[R] = E[V + X] = E[V] + E[X] = 0. \]  \hspace{1cm} (2)

(b) Because \( X \) and \( V \) are independent random variables, the variance of \( R \) is the sum of the variance of \( V \) and the variance of \( X \).

\[ \text{Var}[R] = \text{Var}[V] + \text{Var}[X] = 12 + 3 = 15. \]  \hspace{1cm} (3)

(c) Since \( E[R] = E[V] = 0 \),

\[ \text{Cov}[V, R] = E[VR] = E[V(V + X)] = E[V^2] = \text{Var}[V]. \]  \hspace{1cm} (4)

[Continued]
(d) The correlation coefficient of $V$ and $R$ is
\[ \rho_{V,R} = \frac{\text{Cov}[V, R]}{\sqrt{\text{Var}[V]} \sqrt{\text{Var}[R]}} = \frac{\text{Var}[V]}{\sqrt{\text{Var}[V]} \sqrt{\text{Var}[R]}} = \frac{\sigma_V}{\sigma_R}. \] (5)

The LMSE estimate of $V$ given $R$ is
\[ \hat{V}(R) = \rho_{V,R} \frac{\sigma_V}{\sigma_R} (R - \text{E}[R]) + \text{E}[V] = \frac{\sigma_V^2}{\sigma_R^2} R = \frac{12}{15} R. \] (6)

Therefore $a^* = 12/15 = 4/5$ and $b^* = 0$.

(e) The minimum mean square error in the estimate is
\[ e^* = \text{Var}[V](1 - \rho_{V,R}^2) = 12(1 - 12/15) = 12/5. \] (7)