Show that, for example,
\[ P_1(t) = 12 - 48t^2 + 16t^4 \]
\[ P_2(t) = -120t + 160t^3 - 32t^5 \]

2. Verify that the function \( x(t) = \frac{t^2}{4} \) solves the initial-value problem
\[
\begin{aligned}
x' &= \sqrt{x} \\
x(0) &= 0
\end{aligned}
\]
Apply the Taylor-series method of order 1, and explain why the numerical solution differs from the solution \( \frac{t^2}{4} \).

3. Compute \( x(0.1) \) by solving the differential equation
\[
\begin{aligned}
x' &= -tx^2 \\
x(0) &= 2
\end{aligned}
\]
with one step of the Taylor-series method of order 2. (Use a calculator.)

4. Using the ordinary differential equation
\[
\begin{aligned}
x' &= x^2 + xe' \\
x(0) &= 1
\end{aligned}
\]
and one step of the Taylor-series method of order 3, calculate \( x(0.01) \).

5. Consider the ordinary differential equation
\[
\begin{aligned}
5tx' + x^2 &= 2 \\
x(4) &= 1
\end{aligned}
\]
Calculate \( x(4.1) \) using one step of the Taylor-series method of order 2.

6. An integral equation is an equation involving an unknown function within an integral. For example, here is a typical integral equation (of a type known by the name Volterra):
\[
x(t) = \int_0^t \cos(s + x(s)) \, ds + e^t
\]
By differentiating this integral equation, obtain an equivalent initial-value problem for the unknown function.

7. If the Taylor-series method is used to solve an initial-value problem involving the differential equation
\[ x' = \cos(tx) \]
what are the formulas for \( x'', x''' \), and \( x^{(4)} \)?

8. Let \( x' = f(t, x) \). Determine \( x'', x''' \), and \( x^{(4)} \) from this equation.

**COMPUTER PROBLEMS 8.2**

1. Write and test a computer program to solve the following differential equation with initial condition
on the interval \([1, 3]\). Use the Taylor series of order 5 and \(h = 0.01\).

2. Write and test a computer program for solving the following initial-value problem:

\[
\begin{cases}
x' = 1 + x^2 - t^3 \\
x(0) = -1
\end{cases}
\]

Use the Taylor-series method of order 4, with \(h\) a binary machine number near 0.01. Find the solution in the interval \([0, 2]\). Account for any peculiar phenomenon in the solution.

3. Methods for solving initial-value problems also can be used to compute definite or indefinite integrals. For example, we can compute

\[
\int_0^2 e^{-x^2} \, ds
\]

by solving the initial-value problem

\[
\begin{cases}
x' = e^{-x^2} \\
x(0) = 0
\end{cases}
\]

on the \(t\)-interval \([0, 2]\). Do this, using the Taylor-series method of order 4. From a table of the error function

\[
\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} \, ds
\]

we obtain \(x(2) \approx 0.8820813907\). (See Abramowitz and Stegun [1964, p. 311].)

4. (Continuation) Use Problem 8.2.1 (p. 535) to write a computer program for the Taylor-series method of order 6 applicable to the integral in the preceding computer problem. Test the code with \(h = 0.01\) to see whether the same value of \(x(2)\) is obtained. Print a table of the function \(\frac{1}{2} \sqrt{\pi} \text{erf}(t)\) in steps of 0.01 from \(t = 0\) to \(t = 2\).

5. Solve the initial-value problem

\[
\begin{cases}
x' = 1 + x^2 \\
x(0) = 0
\end{cases}
\]

on the interval \([0, 1.56]\) using the Taylor-series method of order 4 with \(h = 0.01\). Then use the computed value of \(x(1.56)\) as the initial value to integrate back to \(t = 0\). Compare the results and explain what happened.

6. The equation \(\arctan(x/t) = \ln \sqrt{x^2 + t^2}\) defines \(x\) implicitly as a function of \(t\). Verify that this implicit function is a solution of the initial-value problem

\[
\begin{cases}
x' = (t + x)/(t - x) \\
x(1) = 0
\end{cases}
\]

Prepare a table of the function \(x(t)\) on the interval \([0, 2]\) with steps of \(\pm 0.01\). Use the Taylor-series method of order 4.

7. The function

\[
\varphi(t) = \int_0^t \sin s^2 \, ds
\]