THEOREM 6  Theorem on Global Truncation Error Approximation

If the local truncation errors in the numerical solution are $O(h^{m+1})$, then the global truncation error is $O(h^m)$.

Proof  In Theorem 5, on global truncation error bound, let $\delta$ be $O(h^{m+1})$. Since $e^z - 1$ is $O(z)$ and $nh = t$, we find a decrease of $1$ in the order from using the formula in Theorem 5.

PROBLEMS 8.5

1. Discuss these multistep methods in light of Theorem 1 (p. 558), on multistep method stability and consistency:
   a. $x_n - x_{n-2} = 2hf_{n-1}$
   b. $x_n - x_{n-2} = h\left[\frac{7}{6}f_{n-1} - \frac{5}{3}f_{n-2} + \frac{1}{2}f_{n-3}\right]$
   c. $x_n - x_{n-1} = h\left[\frac{19}{12}f_n + \frac{5}{12}f_{n-1} - \frac{5}{12}f_{n-2} + \frac{1}{12}f_{n-3}\right]

2. A method is said to be weakly unstable if $p(z)$ has a zero $\lambda$ such that $\lambda \neq 1$, $|\lambda| = 1$, and $q(\lambda) < \lambda p'(\lambda)$. Show that the Milne method given by Equation (12) is weakly unstable.

3. Show that every multistep method in which $p(z) = z^k - z^{k-1}$ and $\sum_{i=0}^{k} b_i = 1$ is stable, consistent, convergent, and weakly stable.

4. Determine the numerical characteristics of the multistep method whose equation is

   $$x_n + 4x_{n-1} - 5x_{n-2} = h(4f_{n-1} + 2f_{n-2})$$

5. Is there any reason for distrusting this numerical scheme for solving $x' = f(t, x)$?

   $$x_n - 3x_{n-1} + 2x_{n-2} = h[4f_{n-1} + 2f_{n-1} + f_{n-2} - 2f_{n-3}]$$

   Explain.

6. Which of these multistep methods is convergent?
   a. $x_n - x_{n-2} = h(f_n - 3f_{n-1} + 4f_{n-2})$
   b. $x_n - 2x_{n-1} + x_{n-2} = h(f_n - f_{n-1})$
   c. $x_n - x_{n-1} - x_{n-2} = h(f_n - f_{n-1})$
   d. $x_n - 3x_{n-1} + 2x_{n-2} = h(f_n + f_{n-1})$
   e. $x_n - x_{n-2} = h(f_n - 3f_{n-1} + 2f_{n-2})$

7. A multistep method is said to be strongly stable if $p(1) = 0$, $p'(1) \neq 0$, and all other roots of $p$ satisfy the inequality $|z| < 1$. Prove that a strongly stable method is convergent, using Theorem 1, on multistep stability and consistency. Prove also that for any value of $\lambda$, a strongly stable method will solve the problem $z' = \lambda x$, $x(0) = 1$ without introducing any extraneous errors of exponential growth.

8. Prove that

   $$\frac{Ah^{m+1} + O(h^{m+2})}{B - Ch} = \frac{A}{B}h^{m+1} + O(h^{m+2})$$