Optimal Control Theory: Applications to Management Science and Economics

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CHAPTER 1

WHAT IS OPTIMAL CONTROL THEORY?
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• Dynamic Systems: Evolving over time.
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- Time: Discrete or continuous; Optimal way to control a dynamic system.
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- Prerequisites: Calculus, Vectors and Matrices, ODE and PDE.
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• Dynamic Systems: Evolving over time.
• Time: Discrete or continuous; Optimal way to control a dynamic system.
• Prerequisites: Calculus, Vectors and Matrices, ODE and PDE.
• Applications: Production, Finance, Economics, Marketing and others.
Basic Concepts and Definitions

- A dynamic system is described by state equation:

\[ \dot{x}(t) = f(x(t), u(t), t), \quad x(0) = x_0, \]  

where \( x(t) \) is state variable, \( u(t) \) is control variable.

- The control aim is to maximize the objective function:

\[ J = \int_0^T F(x(t), u(t), t) dt + S[x(T), T]. \]  

- Usually the control variable \( u(t) \) will be constrained as follows:

\[ u(t) \in \Omega(t), \quad t \in [0, T], \]
Sometimes, we consider the following constraints:

1. Inequality constraints (mixed)

\[ g(x(t), u(t), t) \geq 0, \quad t \in [0, T]. \]  

2. Constraints involving only state variables (pure)

\[ h(x(t), t) \geq 0, \quad t \in [0, T]. \]  

3. Terminal state

\[ x(T) \in X, \]  

where \( X \) is called the *reachable set* of the state variable at time \( T \).
Example 1.1 A Production-Inventory Model.

We consider the production and inventory storage of a given good in order to meet an exogenous demand at minimum cost.
# The Production-Inventory Model

<table>
<thead>
<tr>
<th>State Variable</th>
<th>( I(t) = ) Inventory Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control Variable</td>
<td>( P(t) = ) Production Rate</td>
</tr>
<tr>
<td>State Equation</td>
<td>( \dot{I}(t) = P(t) - S(t), \ I(0) = I_0 )</td>
</tr>
<tr>
<td>Objective Function</td>
<td>Maximize ( J = \int_0^T -[h(I(t)) + c(P(t))]dt )</td>
</tr>
<tr>
<td>State Constraint</td>
<td>( I(t) \geq 0 )</td>
</tr>
<tr>
<td>Control Constraints</td>
<td>( 0 \leq P_{\text{min}} \leq P(t) \leq P_{\text{max}} )</td>
</tr>
<tr>
<td>Terminal Condition</td>
<td>( I(T) \geq I_{\text{min}} )</td>
</tr>
<tr>
<td>Exogenous Functions</td>
<td>( S(t) = ) Demand Rate</td>
</tr>
<tr>
<td>Parameters</td>
<td>( T = ) Terminal Time</td>
</tr>
<tr>
<td></td>
<td>( I_{\text{min}} = ) Minimum Ending Inventory</td>
</tr>
<tr>
<td></td>
<td>( P_{\text{min}} = ) Minimum Possible Production Rate</td>
</tr>
<tr>
<td></td>
<td>( P_{\text{max}} = ) Maximum Possible Production Rate</td>
</tr>
<tr>
<td></td>
<td>( I_0 = ) Initial Inventory Level</td>
</tr>
</tbody>
</table>
Example 1.2 An Advertising Model.

We consider a special case of the Nerlove-Arrow advertising model.
# The Advertising Model

<table>
<thead>
<tr>
<th><strong>State Variable</strong></th>
<th>$G(t) =$ Advertising Goodwill</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Control Variable</strong></td>
<td>$u(t) =$ Advertising Rate</td>
</tr>
<tr>
<td><strong>State Equation</strong></td>
<td>$\dot{G}(t) = u(t) - \delta G(t), G(0) = G_0$</td>
</tr>
<tr>
<td><strong>Objective Function</strong></td>
<td>Maximize ${ J = \int_0^\infty [\pi(G(t)) - u(t)]e^{-\rho t}dt }$</td>
</tr>
<tr>
<td><strong>State Constraint</strong></td>
<td>$\cdots$</td>
</tr>
<tr>
<td><strong>Control Constraints</strong></td>
<td>$0 \leq u(t) \leq Q$</td>
</tr>
<tr>
<td><strong>Terminal Condition</strong></td>
<td>$\cdots$</td>
</tr>
<tr>
<td><strong>Exogenous Function</strong></td>
<td>$\pi(G) =$ Gross Profit Rate</td>
</tr>
<tr>
<td><strong>Parameters</strong></td>
<td>$\delta =$ Goodwill Decay Constant</td>
</tr>
<tr>
<td></td>
<td>$\rho =$ Discount Rate</td>
</tr>
<tr>
<td></td>
<td>$Q =$ Upper Bound on Advertising Rate</td>
</tr>
<tr>
<td></td>
<td>$G_0 =$ Initial Goodwill Level</td>
</tr>
</tbody>
</table>
Example 1.3 A Consumption Model.

We consider a problem of an agent’s consumption of his wealth over time in a way that maximizes his consumption utility over his lifetime.
## The Consumption Model

<table>
<thead>
<tr>
<th>State Variable</th>
<th>$W(t) = \text{Wealth}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control Variable</td>
<td>$C(t) = \text{Consumption Rate}$</td>
</tr>
<tr>
<td>State Equation</td>
<td>$\dot{W}(t) = rW(t) - C(t),\ W(0) = W_0$</td>
</tr>
<tr>
<td>Objective Function</td>
<td>$\max \left{ J = \int_0^T U(C(t))e^{-\rho t} , dt + B[W(T)]e^{-\rho T} \right}$</td>
</tr>
<tr>
<td>State Constraint</td>
<td>$W(t) \geq 0$</td>
</tr>
<tr>
<td>Control Constraint</td>
<td>$C(t) \geq 0$</td>
</tr>
<tr>
<td>Terminal Condition</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>Exogenous Functions</td>
<td>$U(C) = \text{Utility of Consumption}$</td>
</tr>
<tr>
<td>Parameters</td>
<td>$B(W) = \text{Bequest Function}$</td>
</tr>
<tr>
<td>Parameters</td>
<td>$T = \text{Terminal Time}$</td>
</tr>
<tr>
<td>Parameters</td>
<td>$W_0 = \text{Initial Wealth}$</td>
</tr>
<tr>
<td>Parameters</td>
<td>$\rho = \text{Discount Rate}$</td>
</tr>
<tr>
<td>Parameters</td>
<td>$r = \text{Interest Rate}$</td>
</tr>
</tbody>
</table>
HISTORY OF OPTIMAL CONTROL THEORY

• Calculus of Variations.
• Brachistochrone problem: path of least time
• Newton, Leibniz, Bernoulli brothers, Jacobi, Bolza.
FIGURE 1.1 THE BRACHISTOCHRONE PROBLEM
NOTATION AND CONCEPTS USED

• “=” “is equal to” or “is defined to be equal to” or “is identically equal to”
• “:=” “is defined to be equal to,”
• “≡” “is identically equal to,”
• “≈” “is approximately equal to.”
• “⇒” “implies”
• “∈” “is a member of.”
Let $y$ be an $n$-component column vector and $z$ be an $m$-component row vector, i.e.,

$$
y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = (y_1, \ldots, y_n)^T \quad \text{and} \quad z = (z_1, \ldots, z_m),$$

$\dot{y} := dy/dt$ and $\dot{z} := dz/dt$ are defined as

$$\dot{y} = \frac{dy}{dt} = (\dot{y}_1, \ldots, \dot{y}_n)^T \quad \text{and} \quad \dot{z} = \frac{dz}{dt} = (\dot{z}_1, \ldots, \dot{z}_m),$$

When $n = m$, we can define the inner product

$$zy = \sum_{i=1}^{n} z_i y_i.$$
More generally, if $A = \{a_{ij}\} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mk} \end{pmatrix}$ is an $m \times k$ matrix and $B = \{b_{ij}\}$ is a $k \times n$ matrix, we define the matrix product $C = \{c_{ij}\} = AB$, which is an $m \times n$ matrix with components

$$c_{ij} = \sum_{r=1}^{k} a_{ir} b_{rj}. \quad (8)$$
DIFFERENTIATING VECTORS AND MATRICES
W.R.T. SCALARS

• Let \( f : E^1 \to E^k \) be a \( k \)-dimensional function of a scalar variable \( t \). If \( f \) is a row vector, then

\[
\frac{df}{dt} = f_t = (f_{1t}, f_{2t}, \ldots, f_{kt}), \text{ a row vector.}
\]

• If \( f \) is a column vector, then

\[
\frac{df}{dt} = f_t = \begin{pmatrix}
  f_{1t} \\
  f_{2t} \\
  \vdots \\
  f_{kt}
\end{pmatrix}
= (f_{1t}, f_{2t}, \ldots, f_{kt})^T, \text{ a column vector.}
\]
If \( F(y, z) \) is a scalar function of \( n \)-dimensional column vector \( y \) and \( m \)-dimensional row vector \( z \), \( n \geq 2 \), \( m \geq 2 \), then the gradients \( F_y \) and \( F_z \) are defined, respectively, as

\[
F_y = (F_{y_1}, \cdots, F_{y_n}), \quad \text{a row vector,}
\]

and

\[
F_z = \begin{pmatrix}
F_{z_1} \\
\vdots \\
F_{z_m}
\end{pmatrix}, \quad \text{a column vector,}
\]
If \( f : E^n \times E^m \rightarrow E^k \) is a \( k \)-dimensional vector function \( f \) either row or column, \( k \geq 2 \), i.e.,

\[
f = (f_1, \cdots, f_k) \text{ or } f = (f_1, \cdots, f_k)^T,
\]

where \( f_i = f_i(y, z) \) depends on column \( y \in E^n \) and row \( z \in E^m \), \( n \geq 2, m \geq 2 \), then \( f_z \) denotes \( k \times m \) matrix:

\[
f_z = \begin{pmatrix}
\frac{\partial f_1}{\partial z_1}, & \frac{\partial f_1}{\partial z_2}, & \cdots & \frac{\partial f_1}{\partial z_m} \\
\frac{\partial f_2}{\partial z_1}, & \frac{\partial f_2}{\partial z_2}, & \cdots & \frac{\partial f_2}{\partial z_m} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_k}{\partial z_1}, & \frac{\partial f_k}{\partial z_2}, & \cdots & \frac{\partial f_k}{\partial z_m}
\end{pmatrix} = \{\frac{\partial f_i}{\partial z_j}\},
\]

(11)
$f_y$ will denote the $k \times n$ matrix

\[
\begin{pmatrix}
\frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \cdots & \frac{\partial f_1}{\partial y_n} \\
\frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \cdots & \frac{\partial f_2}{\partial y_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_k}{\partial y_1} & \frac{\partial f_k}{\partial y_2} & \cdots & \frac{\partial f_k}{\partial y_n}
\end{pmatrix}
= \{\partial f_i/\partial y_j\}.
\]

Matrices $f_z$ and $f_y$ are known as Jacobian matrices. Note that

\[
f_z = f_z^T = f_{zT} = f_{zT}^T,
\]

where by $f_z^T$ we mean $(f^T)_z$ and not $(f_z)^T$. 

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Applying the rule (11) to \( F_y \) in (9) and the rule (12) to \( F_z \) in (10), respectively, we obtain \( F_{yz} = (F_y)_z \) to be the \( n \times m \) matrix

\[
F_{yz} = \begin{pmatrix}
F_{y1z1} & F_{y1z2} & \cdots & F_{y1zm} \\
F_{y2z1} & F_{y2z2} & \cdots & F_{y2zm} \\
\vdots & \vdots & \ddots & \vdots \\
F_{ynz1} & F_{ynz2} & \cdots & F_{ynzm}
\end{pmatrix} = \left\{ \frac{\partial^2 F}{\partial y_i \partial z_j} \right\}, \quad (13)
\]
and $F_{zy} = (F_z)_y$ to be the $m \times n$ matrix

$$F_{zy} = \begin{pmatrix}
F_{z_1 y_1} & F_{z_1 y_2} & \cdots & F_{z_1 y_n} \\
F_{z_2 y_1} & F_{z_2 y_2} & \cdots & F_{z_2 y_n} \\
\vdots & \vdots & \ddots & \vdots \\
F_{z_m y_1} & F_{z_m y_2} & \cdots & F_{z_m y_n}
\end{pmatrix} = \left\{ \frac{\partial^2 F}{\partial z_i \partial y_j} \right\}. \quad (14)$$
Let $g$ be an $n$-component row vector function and $f$ be an $n$-component column vector function of an $n$-component vector $x$, Then

$$(gf)_x = gf_x + f^T g_x^T = gf_x + f^T g_x.$$  \hspace{1cm} (15)$$

If $g = F_x$, then

$$(gf)_x = (F_x f)_x = F_x f_x + f^T F_{xx} = F_x f_x + (F_{xx} f)^T.$$  \hspace{1cm} (16)$$
• The norm of an \( m \)-component row or column vector \( z \) is defined to be

\[
\| z \| = \sqrt{z_1^2 + \cdots + z_m^2}.
\]  

(17)

• Neighborhood \( N_{z_0} \) of a point, e.g.,

\[
N_{z_0} = \{ z \mid \| z - z_0 \| < \varepsilon \},
\]

(18)

where \( \varepsilon > 0 \) is a small positive real number.
**Little-o Notation and Norm of a Function**

- A function $F(z) : E^m \rightarrow E^1$ is said to be of the order $o(z)$, if
  \[ \lim_{\|z\| \rightarrow 0} \frac{F(z)}{\|z\|} = 0. \]

- The *norm* of an $m$-dimensional row or column vector function $z(t), t \in [0, T]$, is defined to be
  \[ \|z\| = \left[ \sum_{j=1}^{m} \int_{0}^{T} z_j^2(\tau) d\tau \right]^{\frac{1}{2}}. \quad (19) \]
Some Special Notation

• The concepts of left and right limits

\[ x(t^{-}) = \lim_{\varepsilon \downarrow 0} x(t - \varepsilon) \quad \text{and} \quad x(t^{+}) = \lim_{\varepsilon \downarrow 0} x(t + \varepsilon). \quad (20) \]

• Discrete-time models (in Chapter 8-9)

\[ x^k: \text{ state variable at time } k. \]
\[ u^k: \text{ control variable at time } k. \]
\[ \lambda^k: \text{ adjoint variables, respectively, at time } k, \]
\[ \Delta x^k := x^{k+1} - x^k: \text{ difference operator.} \]
\[ x^{k*} \text{ and } u^{k*}: \text{ quantities along an optimal path.} \]
Some Special Notation cont.

- **bang function**

\[
\text{bang}[b_1, b_2; W] = \begin{cases} 
  b_1 & \text{if } W < 0, \\
  \text{undefined} & \text{if } W = 0, \\
  b_2 & \text{if } W > 0.
\end{cases}
\]  

(21)

- **Sat function**

\[
\text{sat}[y_1, y_2; W] = \begin{cases} 
  y_1 & \text{if } W < y_1, \\
  W & \text{if } y_1 \leq W \leq y_2, \\
  y_2 & \text{if } W > y_2.
\end{cases}
\]  

(22)
Impulse Control

\[ \text{imp}(x_1, x_2, t) = \lim_{\varepsilon \to 0} \int_{t}^{t+\varepsilon} F(x, u, \tau) d\tau. \]  

(23)

If the impulse is applied only at time \( t \), then we can calculate the objective function as

\[ J = \int_{0}^{t} F(x, u, \tau) d\tau + \text{imp}(x_1, x_2, t) + \int_{0}^{T} F(x, u, \tau) d\tau + S[x(T), T]. \]

(24)
A set $D \subset E^n$ is a *convex set* if for each pair of points $y, z \in D$, the entire line segment joining these two points is also in $D$, i.e.,
\[ py + (1 - p)z \in D, \quad \text{for each } p \in [0, 1]. \]

Given $x^i \in E^n, i = 1, 2, \ldots, l$, we define $y \in E^n$ to be a *convex combination* of $x^i \in E^n$, if there exists $p_i \geq 0$ such that
\[
\sum_{i=1}^{l} p_i = 1 \quad \text{and} \quad y = \sum_{i=1}^{l} p_i x^i.
\]
The convex hull of a set $D \subset E^n$ is

$$\text{co}D := \left\{ \sum_{i=1}^{l} p_i x^i : \sum_{i=1}^{l} p_i = 1, \ p_i \geq 0, \ x^i \in D, \ i = 1, 2, \ldots, l \right\}.$$
\[ \psi : D \to E^1, \text{ is } \text{concave}, \text{ if for each pair of points } y, z \in D \text{ and for all } p \in [0, 1], \]
\[ \psi(py + (1 - p)z) \geq p\psi(y) + (1 - p)\psi(z). \]

• If \( \geq \) is changed to \( > \) for all \( y, z \in D \) with \( y \neq z \), and \( 0 < p < 1 \), then \( \psi \) is called a strictly concave function.

• If \( \psi(x) \) is a differentiable function on the interval \([a, b]\), then it is concave, if for each pair of points \( y, z \in [a, b] \),
\[ \psi(z) \leq \psi(y) + \psi_x(y)(z - y). \]
**Figure 1.2 A Concave Function**

![Diagram of a concave function](image)

- **Axes:** $\psi(x)$ and $x$
- **Points:** $A$, $B$, $a$, $y$, $z$, $b$, and $D$
- **Lines:** Line segment joining $A$ and $B$
CONCAVE AND CONVEX FUNCTIONS

• If the function $\psi$ is twice differentiable, then it is \textit{concave}, if at each point in $[a, b],$

$$\psi_{xx} \leq 0.$$  

In case $x$ is a vector, $\psi_{xx}$ needs to be a negative definite matrix.

• If $\psi : D \rightarrow E^1$ defined on a convex set $D \subset E^n$ is a concave function, then the negative of the function $\psi$, i.e., $-\psi : D \rightarrow E^1$, is a \textit{convex function}. 
AFFINE AND HOMOGENEOUS FUNCTIONS

• $\psi : E^n \to E^1$ is said to be **affine**, if $\psi(x) - \psi(0)$ is linear.

• $\psi : E^n \to E^1$ is said to be **homogeneous of degree one**, if $\psi(bx) = b\psi(x)$, where $b$ is a scalar constant.

• **Saddle Point** $\psi : E^n \times E^m \to E^1$. For $\max_x \min_y$, a point $(\hat{x}, \hat{y}) \in E^n \times E^m$ is called a **saddle point** of $\psi(x, y)$, if

$$\psi(x, \hat{y}) \leq \psi(\hat{x}, \hat{y}) \leq \psi(\hat{x}, y) \text{ for all } x \in E^n \text{ and } y \in E^m.$$ 

Note also that

$$\psi(\hat{x}, \hat{y}) = \max_x \psi(x, \hat{y}) = \min_y \psi(\hat{x}, y).$$

Similarly, for $\min_x \max_y$. Just reverse the inequalities.
Figure 1.3 A Saddle Point
A set of vectors $a_1, a_2, \ldots, a_n$ from $E^n$ is said to be *linearly dependent* if there exist scalars $p_i$ not all zero such that

$$\sum_{i=1}^{n} p_i a_i = 0.$$  \hfill (25)

If the only set of $p_i$ for which (1.25) holds is $p_1 = p_2 = \cdots = p_n = 0$, then the vectors are said to be *linearly independent*. 

*Linear Independence and Rank of a Matrix*
Linear Independence and Rank of a Matrix

The *rank* (or more precisely the column rank) of an $m \times n$ matrix $A$, written $\text{rank}(A)$, is the maximum number of linearly independent columns in $A$.

An $m \times n$ matrix is of *full rank* if

$$\text{rank}(A) = n.$$