and $S$ is linear, i.e.,

$$
\begin{align*}
\max_{u(t) \in \Omega(t)} \{ J = cx(T) \} \\
\text{subject to} \\
\dot{x} = f(x, u, t), \ x(0) = x_0,
\end{align*}
$$

where $c = (c_1, c_2, \cdots, c_n)$ is an $n$-dimensional row vector of given constants. In the next paragraph and in Exercise 2.3, it will be demonstrated that all of these forms can be converted into the linear Mayer form.

To show that the Bolza form can be reduced to the linear Mayer form, we define a new state vector $y = (y_1, y_2, \ldots, y_{n+1})$, having $n + 1$ components defined as follows: $y_i = x_i$ for $i = 1, \ldots, n$ and

$$
\dot{y}_{n+1} = F(x, u, t) + \frac{\partial S(x, t)}{\partial x} f(x, u, t) + \frac{\partial S(x, t)}{\partial t}, \ y_{n+1}(0) = S(x_0, 0).
$$

(2.6)

We also put $c = (0, \cdots, 0, 1)$, where $c$ has $n + 1$ components, so that the objective function is $J = cy(T) = y_{n+1}(T)$. If we now integrate (2.6) from 0 to $T$, we have

$$
J = cy(T) = y_{n+1}(T) = \int_0^T F(x, u, t) dt + S[x(T), T],
$$

(2.7)

which is the same as the objective function $J$ in (2.4). Of course, the price paid for going from Bolza to linear Mayer form is the addition of one state variable and its associated differential equation (2.6).

Exercise 2.3 poses the question of showing in a similar way that the Lagrange and Mayer forms can also be reduced to the linear Mayer form.

In Section 2.2, we derive necessary conditions for optimal control in the form of the maximum principle, and in Section 2.4 we derive sufficient conditions. In any particular application, the existence of a solution will be demonstrated by actually finding a solution that satisfies both the necessary and the sufficient conditions for optimality. We thus avoid the necessity of having to prove general existence theorems, which require advanced and difficult mathematics. Nevertheless, interested readers can consult Hartl, Sethi, and Vickson (1995) for a brief discussion of existence results and references therein including Cesari (1983) for further details.
2.2 Dynamic Programming and the Maximum Principle

We shall now derive the maximum principle by using a dynamic programming approach. The proof is intuitive in nature and is not intended to be mathematically rigorous. For more rigorous derivations, we refer the reader to Appendix C, Pontryagin et al. (1962), Berkovitz (1961), Halkin (1967), Boltyanskii (1971), Hartberger (1973), Bryant and Mayne (1974), Leitmann (1981), and Seierstad and Sydsæter (1987). Additional references can be found in the survey by Hartl, Sethi, and Vickson (1995). For maximum principles for more general optimal control problems including those with nondifferentiable functions, see Clarke (1976, 1983, 1989).

2.2.1 The Hamilton-Jacobi-Bellman Equation

Suppose $V(x, t) : E^n \times E^1 \rightarrow E^1$ is a function whose value is the maximum value of the objective function of the control problem for the system, given that we start it at time $t$ in state $x$. That is,

$$V(x, t) = \max_{u(s) \in \Omega(s)} \left[ \int_t^T F(x(s), u(s), s)ds + S(x(T), T) \right], \quad (2.8)$$

where for $s \geq t$,

$$\frac{dx}{ds} = f(x(s), u(s), s), \quad x(t) = x.$$

We initially assume that the value function $V(x, t)$ exists for all $x$ and $t$ in the relevant ranges. Later we will make additional assumptions about the function $V(x, t)$.

Richard Bellman (1957) in his book on dynamic programming states the **principle of optimality** as follows:

An optimal policy has the property that, whatever the initial state and initial decision are, the remaining decision must constitute an optimal policy with regard to the outcome resulting from the first decision.

Intuitively this principle is obvious, for if we were to start in state $x$ at time $t$ and did not follow an optimal path from there on, then there would exist (by assumption) a better path from $t$ to $T$, hence we could
where $\delta t$ represents a small increment in $t$. It is instructive to compare this equation to definition (2.8).

Since $F$ is a continuous function, the integral in (2.9) is approximately $F(x,u,t)\delta t$ so that we can rewrite (2.9) as

$$V(x,t) = \max_{u \in \Omega(t)} \{ F(x,u,t)\delta t + V[x(t + \delta t), t + \delta t] \} + o(\delta t), \quad (2.10)$$

where $o(\delta t)$ denotes a collection of higher-order terms in $\delta t$. (By definition given in Section 1.4, $o(\delta t)$ is a function such that $\lim_{\delta t \to 0} \frac{o(\delta t)}{\delta t} = 0$).

We now make an assumption of which we will talk more later. We assume that the value function $V$ is a continuously differentiable function of its arguments. This allows us to use the Taylor series expansion of $V$ with respect to $\delta t$ and obtain

$$V[x(t + \delta t), t + \delta t] = V(x,t) + [V_x(x,t)\dot{x} + V_t(x,t)]\delta t + o(\delta t), \quad (2.11)$$

where $V_x$ and $V_t$ are partial derivatives of $V(x,t)$ with respect to $x$ and $t$, respectively.

Substituting for $\dot{x}$ from (2.1) in the above equation and then using it in (2.10), we obtain

$$V(x,t) = \max_{u \in \Omega(t)} \{ F(x,u,t)\delta t + V(x,t) + V_x(x,t)f(x,u,t)\delta t + V_t(x,t)\delta t \} + o(\delta t). \quad (2.12)$$

Canceling $V(x,t)$ on both sides and then dividing by $\delta t$ we get

$$0 = \max_{u \in \Omega(t)} \{ F(x,u,t) + V_x(x,t)f(x,u,t) + V_t(x,t) \} + \frac{o(\delta t)}{\delta t}. \quad (2.13)$$

Now we let $\delta t \to 0$ and obtain the following equation

$$0 = \max_{u \in \Omega(t)} \{ F(x,u,t) + V_x(x,t)f(x,u,t) + V_t(x,t) \}, \quad (2.14)$$

for which the boundary condition is

$$V(x,T) = S(x,T). \quad (2.15)$$

That this boundary condition must hold follows from the fact that at $t = T$, the value function is simply the salvage value function.
The components of the vector $V_x(x, t)$ can be interpreted as the marginal contributions of the state variables $x$ to the maximized objective function (2.8). We denote the marginal return vector (along the optimal path $x^*(t)$) by the adjoint (row) vector $\lambda(t) \in E^n$, i.e.,

$$\lambda(t) = V_x(x^*(t), t) := V_x(x, t) \big|_{x=x^*(t)} \quad (2.16)$$

From the preceding remark, we can also interpret $\lambda(t)$ as the per unit change in the objective function for a small change in $x^*(t)$ at time $t$; see Section 2.2.4. Next we introduce the so-called Hamiltonian

$$H[x, u, V_x, t] = F(x, u, t) + V_x(x, t) f(x, u, t) \quad (2.17)$$

or, simply,

$$H(x, u, \lambda, t) = F(x, u, t) + \lambda f(x, u, t) \quad (2.18)$$

We can rewrite equation (2.14) as the following equation,

$$0 = \max_{u \in \Omega(t)} [H(x, u, V_x, t) + V_t] \quad (2.19)$$

called the Hamilton-Jacobi-Bellman equation or, simply, the HJB equation.

Note that it is possible to take $V_t$ out of the maximizing operation since it does not depend on $u$.

The Hamiltonian maximizing condition of the maximum principle can be obtained from (2.19) and (2.16) by observing that, if $x^*(t)$ and $u^*(t)$ are optimal values of the state and control variables and $\lambda(t)$ is the corresponding value of the adjoint variable at time $t$, then the optimal control $u^*(t)$ must satisfy (2.19), i.e., for all $u \in \Omega(t),

$$H[x^*(t), u^*(t), \lambda(t), t] + V_t(x^*(t), t) \geq H[x^*(t), u, \lambda(t), t] + V_t(x^*(t), t) \quad (2.20)$$

Canceling the term $V_t$ on both sides, we obtain

$$H[x^*(t), u^*(t), \lambda(t), t] \geq H[x^*(t), u, \lambda(t), t] \quad (2.21)$$

for all $u \in \Omega(t)$.

In order to complete the statement of the maximum principle, we must still obtain the adjoint equation.
Rewriting the first equation as

\[-d\lambda = H_x dt = F_x dt + \lambda f_x dt,\]

we can observe that along the optimal path, \(-d\lambda\), the negative of the increase in the price of capital from \(t\) to \(t + dt\), which can be considered as the marginal cost of holding that capital, equals the marginal revenue \(H_x dt\) of investing the capital. In turn the marginal revenue, \(H_x dt\), consists of the sum of direct marginal contribution, \(F_x dt\), and the indirect marginal contribution, \(\lambda f_x dt\). Thus, the adjoint equation becomes the equilibrium relation—marginal cost equals marginal revenue, which is a familiar concept in the economics literature. See, e.g., Cohen and Cyert (1965, p.189) or Takayama (1974, p.712).

Further insight can be obtained by integrating the above adjoint equation from \(t\) to \(T\) as follows:

\[
\lambda(t) = \lambda(T) + \int_t^T H_x(x(\tau), u(\tau), \lambda(\tau), \tau) d\tau = S_x[x(T), T] + \int_t^T H_x d\tau.
\]

Note that the price \(\lambda(T)\) of a unit of capital at time \(T\) is its marginal salvage value, \(S_x[x(T), T]\). The price \(\lambda(t)\) of a unit of capital at time \(t\) is the sum of its terminal price, \(\lambda(T)\), plus the integral of the marginal surrogate profit rate, \(H_x\), from \(t\) to \(T\).

The above interpretations show that the adjoint variables behave in much the same way as do the dual variables in linear (and nonlinear) programming. The differences being that here the adjoint variables are time dependent and satisfy derived differential equations. These connections will become clearer in Chapter 8, which addresses the discrete maximum principle.

### 2.3 Elementary Examples

In order to absorb the maximum principle, the reader should study very carefully the examples in this section, all of which are problems having only one state and one control variable. Some or all of the exercises at the end of the chapter should also be worked.

In the following examples and others in this book, we shall omit the superscript * on the optimal value of the state variable when there is no confusion arising in doing so.
2.4. Sufficiency Conditions

under which the maximum principle conditions are also sufficient for optimality. This theorem is important from our point of view since the models derived from many management science applications will satisfy conditions required for the sufficiency result. As remarked earlier, our technique for proving existence will be to display for any given model, a solution that satisfies both necessary and sufficient conditions. A good reference for sufficiency conditions is Seierstad and Sydsæter (1987).

We first define a function $H^0 : \mathbb{E}^n \times \mathbb{E}^m \times \mathbb{E}^1 \rightarrow \mathbb{E}^1$ called the derived Hamiltonian as follows:

$$H^0(x, \lambda, t) = \max_{u \in \Omega(t)} H(x, u, \lambda, t). \quad (2.61)$$

We assume that by this equation a function $u = u^0(x, \lambda, t)$ is implicitly and uniquely defined. Given these assumptions we have by definition,

$$H^0(x, \lambda, t) = H(x, u^0, \lambda, t). \quad (2.62)$$

It is also possible to show that

$$H^0_x(x, \lambda, t) = H_x(x, u^0, \lambda, t). \quad (2.63)$$

To see this for the case of differentiable $u^0$, let us differentiate (2.62) with respect to $x$:

$$H^0_x(x, \lambda, t) = H_x(x, u^0, \lambda, t) + H_u(x, u^0, \lambda, t) \frac{\partial u^0}{\partial x}. \quad (2.64)$$

For the second term on the right-hand side of (2.64), we show

$$H_u(x, u^0, \lambda, t) \frac{\partial u^0}{\partial x} = 0 \quad (2.65)$$

for all $x$. There are three cases to consider for the unconstrained global maximum of $H$. (i) It is in the interior of $\Omega(t)$. Here $H_u(x, u^0, \lambda, t) = 0$. (ii) It occurs outside of $\Omega(t)$. Here $\partial u^0/\partial x = 0$, because changing $x$ slightly does not influence the optimal value of $u$. (iii) It is on the boundary of $\Omega(t)$. Here for each $i, j$, either $H_{u_i} = 0$ or $\partial u^0/\partial x_j = 0$ or both. Thus (2.65) and, therefore, (2.63) hold. Exercise 2.15 gives a specific instance of this case.

Remark 2.1. We have shown the result in (2.63) for cases where $u^0$ is a differentiable function of $x$. It holds more generally provided $\Omega(t)$ is appropriately qualified; see Derzko, Sethi, and Thompson (1984).
where \( J(u) \) is the value of the objective function associated with a control \( u \). Since \( x^*(0) = x(0) = x_0 \), the initial condition, and since \( \lambda(T) = S_2[x^*(T), T] \) from the terminal adjoint condition in (2.32), we have

\[
J(u^*) \geq J(u) \tag{2.74}
\]

Thus, \( u^* \) is an optimal control. This completes the proof. \( \Box \)

Note that if the problem is given in the Lagrange form, i.e., \( S(x, T) \equiv 0 \), then all we need is the concavity of \( H^0(x, \lambda, t) \) in \( x \) for each \( t \). Since \( \lambda(t) \) is not known \textit{a priori}, it is usual to test \( H^0 \) for a stronger assumption, i.e., to check for the concavity of the function \( H^0(x, \lambda(t), t) \) in \( x \) for any \( \lambda \) and \( t \). Sometimes the stronger condition given in Exercise 2.19 can be used.

**Example 2.6** Let us show that the problems in Examples 2.1 and 2.2 satisfy the sufficient conditions. We have from (2.36) and (2.61),

\[
H^0 = -x + \lambda u^0,
\]

where \( u^0 \) is given by (2.37). Since \( u^0 \) is a function of \( \lambda \) only, \( H^0(x, \lambda, t) \) is certainly concave in \( x \) for any \( t \) and \( \lambda \) (and in particular for \( \lambda(t) \) supplied by the maximum principle). Since \( S(x, T) = 0 \), the sufficient conditions hold.

Finally, it is important to mention that thus far in this chapter, we have considered problems in which the terminal values of the state variables are not constrained. Such problems are called \textit{free-end-point problems}. The problems at the other extreme, where the terminal values of the state variables are completely specified, are termed \textit{fixed-end-point problems}. Then, there are problems in between these two extremes. While a detailed discussion of terminal conditions on state variables appears in Section 3.4 of the next chapter, it is instructive here to briefly indicate how the maximum principle needs to be modified in the case of fixed-end-point problems. Suppose \( x(T) \) is completely specified, i.e., \( x(T) = k \in \mathbb{R}^n \), where \( k \) is a vector of constants. Observe then that the first term on the right-hand side of inequality (2.73) vanishes regardless of the value of \( \lambda(T) \), since \( x(T) - x^*(T) = k - k = 0 \) in this case. This means that the sufficiency result would go through for any value of \( \lambda(T) \). Not surprisingly, therefore, the transversality condition (2.29) in the fixed-end-point case changes to

\[
\lambda(T) = \beta, \tag{2.75}
\]
where $\beta \in E^n$ is a vector of constants to be determined. The maximum principle for fixed-end-point problems can be restated as (2.32) with $x(T) = k$ and without $\lambda(T) = S_x [x^*(T), T]$. The resulting two-point boundary value problem has initial and final values on the state variables, whereas both initial and terminal values for the adjoint variables are unspecified, i.e., $\lambda(0)$ and $\lambda(T)$ are constants to be determined.

In Exercises 2.9 and 2.18, you are asked to solve the fixed-end-point problems given there.

### 2.5 Solving a TPBVP by Using Spreadsheet Software

A number of examples and exercises found in the rest of this book involve finding a numerical solution to a two-point boundary value problem (TPBVP). In this section we shall show how the GOAL SEEK function in the EXCEL spreadsheet software can be used for this purpose. We will solve the following example.

**Example 2.7** Consider the problem:

$$
\max \left\{ J = \int_{0}^{1} -\frac{1}{2}(x^2 + u^2) dt \right\}
$$

subject to

$$
\dot{x} = -x^3 + u, \ x(0) = 5. \quad (2.76)
$$

**Solution.** We form the Hamiltonian

$$
H = -\frac{1}{2}(x^2 + u^2) + \lambda(-x^3 + u),
$$

where the adjoint variable $\lambda$ satisfies the equation

$$
\dot{\lambda} = x + 3x^2\lambda, \ \lambda(1) = 0. \quad (2.77)
$$

Since $u$ is unconstrained, we set $H_u = 0$ to obtain $u^* = \lambda$. With this, the state equation (2.76) becomes

$$
\dot{x} = -x^3 + \lambda, \ x(0) = 5. \quad (2.78)
$$

Thus, the TPBVP is given by the system of equations (2.77) and (2.78).
In order to solve these equations we discretize them by replacing \( \frac{dx}{dt} \) and \( \frac{d\lambda}{dt} \) by
\[
\frac{\Delta x}{\Delta t} = \frac{x(t + \Delta t) - x(t)}{\Delta t} \quad \text{and} \quad \frac{\Delta \lambda}{\Delta t} = \frac{\lambda(t + \Delta t) - \lambda(t)}{\Delta t},
\]
respectively. Substitution of \( \frac{\Delta x}{\Delta t} \) for \( \dot{x} \) in (2.78) and \( \frac{\Delta \lambda}{\Delta t} \) for \( \dot{\lambda} \) in (2.77) gives the discrete version of the TPBVP:
\[
x(t + \Delta t) = x(t) + [-x(t)^3 + \lambda(t)] \Delta t, \quad x(0) = 5, \tag{2.79}
\]
\[
\lambda(t + \Delta t) = \lambda(t) + [x(t) + 3x(t)^2\lambda(t)] \Delta t, \quad \lambda(1) = 0. \tag{2.80}
\]

In order to solve these equations, open an empty spreadsheet, choose the unit of time to be \( \Delta t = 0.01 \), make a guess for the initial value \( \lambda(0) \) to be, say \(-0.2\), and make the entries in the cells of the spreadsheet as specified below:

- Enter \(-0.2\) in cell A1.
- Enter 5 in cell B1.
- Enter \(= A1 + (B1 + 3 * (B1^2) * A1) * 0.01\) in cell A2.
- Enter \(= B1 + (-B1^3 + A1) * 0.01\) in cell B2.

Note that \( \lambda(0) = -0.2 \) shown as the entry \(-0.2\) in cell A1 is merely a guess. The correct value will be determined by the use of the GOAL SEEK function.

Next highlight cells A2 and B2 and drag the combination down to row 101 of the spreadsheet. Using EDIT in the menu bar, select FILL DOWN. Thus, EXCEL will solve equations (2.80) and (2.79) from \( t = 0 \) to \( t = 1 \) in steps of \( \Delta t = 0.01 \), and that solution will appear as entries in columns A and B of the spreadsheet, respectively. In other words, the guessed solution for \( \lambda(t) \) will appear in cells A1 to A101 and the corresponding solution for \( x(t) \) will appear in cells B1 to B101. To find the correct value for \( \lambda(0) \), use the GOAL SEEK function under TOOLS in the menu bar and make the following entries:

- Set cell: A101.
- To value: 0.

It finds the correct initial value for the adjoint variable as \( \lambda(0) = -0.10437 \), which should appear in cell A1, and the correct ending value
2.5 In Example 2.3, show that in view of (2.47) any \( \lambda(t), t \in [0, 1] \), that satisfies (2.50) must be nonnegative.

2.6 Show that the derived Hamiltonians \( H^0 \) found in Examples 2.3 and 2.5 satisfy the concavity condition required for the sufficiency result in Section 2.4.

2.7 Show that the optimal control obtained from the application of the maximum principle satisfies the principle of optimality: if \( u^*(t) \) is an optimal control and \( x^*(t) \) is the corresponding optimal path for \( 0 \leq t \leq T \) with \( x(0) = x_0 \), then verify the above proposition by showing that \( u^*(t) \) for \( \tau \leq t \leq T \) satisfies the maximum principle for the problem beginning at time \( \tau \) with the initial condition \( x(\tau) = x^*(\tau) \).

2.8 Use the maximum principle to solve the following problem given in the Mayer form:

\[
\max \left[ 8x_1(18) + 4x_2(18) \right]
\]
subject to

\[
\dot{x}_1 = x_1 + x_2 + u, \quad x_1(0) = 15,
\]

\[
\dot{x}_2 = 2x_1 - u, \quad x_2(0) = 20,
\]

and the control constraint

\[
0 \leq u \leq 1.
\]

[Hint: Use the method in Appendix A to solve the simultaneous differential equations.]

2.9 A simple controlled dynamical system is modeled by the scalar equation

\[
\dot{x} = x + u.
\]

The fixed-end-point optimal control problem consists in steering \( x(t) \) from an initial state \( x(0) = x_0 \) to the target \( x(1) = 0 \), such that

\[
J(u) = \frac{1}{4} \int_0^1 u^4 dt
\]

is minimized. Use the maximum principle to show that the optimal control is given by

\[
u^*(t) = \frac{4x_0}{3} \left(e^{-4/3} - 1\right)^{-1} e^{-t/3}.
\]
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Figure 2.7: Water Reservoir of Example 2.12

where \( u(t) \) denotes the net inflow at time \( t \) and \( 0 \leq u \leq 3 \).

Note that \( x(t) \) also represents the water pressure in appropriate units. Since high water pressure is useful for fire-fighting, the objective function in (a) below involves keeping the average pressure high, while that in (b) involves building up a high pressure at \( T = 100 \).

(a) Find the optimal control which maximizes

\[
\int_0^{100} x \, dt.
\]

Find the maximum level reached.

(b) Replace the objective function in (a) by

\[
J = 5x(100),
\]

and re-solve the problem.

(c) Redo the problem with \( J = \int_0^{100} (x - 5u) \, dt \).

2.13 A Machine Maintenance Problem. Consider the machine state dynamics

\[
\dot{x} = -\delta x + u, \quad x(0) = x_0 > 0,
\]

where \( \delta > 0 \) is the rate of deterioration of the machine state and \( u \) is the rate of machine maintenance. Find the optimal maintenance rate so as to

maximize \[
\left\{ J = \int_0^T e^{-\rho t} \left( \pi x - \frac{u^2}{2} \right) \, dt + e^{-\rho T} S x(T) \right\},
\]

\( S \)
where \( \pi > 0 \) with \( \pi x \) representing the profit rate when the machine state is \( x \), \( u^2/2 \) is the cost of maintaining the machine at rate \( u \), \( \rho > 0 \) is the discount rate, \( T \) is the time horizon, and \( S > 0 \) is the salvage value of the machine for each unit of the machine state at time \( T \). Furthermore, show that the optimal maintenance rate decreases, increases, or remains constant over time depending on whether the difference \( S - \pi / (\rho + \delta) \) is negative, positive, or zero, respectively.

2.14 (a) Solve the optimal consumption problem of Example 1.3 with \( U(C) = \ln C \) and \( B = 0 \).

[Hint: Since \( C(t) \geq 0 \), we can replace the state constraint \( W(t) \geq 0 \) by the terminal condition \( W(T) = 0 \), and then use the transversality condition given in (2.75)]

(b) Find the rate of change of optimal consumption over time and conclude that consumption remains constant when \( r = \rho \), increases when \( r > \rho \), and decreases when \( r < \rho \).

2.15 Suppose \( H(x, u, \lambda, t) = \lambda ux - \frac{1}{2}u^2 \) and \( \Omega(t) = [0, 1] \) for all \( t \).

(a) Show that the optimal control \( u^* \) is given by

\[
 u^*(x) = \text{sat} \, [0, 1; \lambda x] = \begin{cases} 
 \lambda x & \text{if } 0 \leq \lambda x \leq 1, \\
 1 & \text{if } \lambda x > 1, \\
 0 & \text{if } \lambda x < 0.
\end{cases}
\]

(b) Verify that (2.63) holds for all values of \( x \) and \( \lambda \).

2.16* Provide an alternative derivation of the adjoint equation in Section 2.2.2 by starting with a restatement of the equation (2.19) as \( -V_t = H^0 \) and differentiating it with respect to \( x \).

2.17 (a) State the two-point boundary value problem (TPBVP) whose solution will give the optimal control \( u^*(t) \) and trajectory \( x^*(t) \) for the problem in Example 2.7, but with a new initial condition \( x(0) = 1 \).

(b) Solve the TPBVP by using a spreadsheet software such as EXCEL.