We present a periodic review inventory model with multiple delivery modes. While base-stock policies are optimal for one or two consecutive delivery modes, they are not so otherwise. For multiple consecutive delivery modes, we show that only the fastest two modes have optimal base stocks, and provide a simple counterexample to show that the remaining ones do not. We investigate why the base-stock policy is or is not optimal in different situations. This paper is an abridged version of Feng et al. (2004).

Subject Classifications: Inventory/production: uncertainty, multiple delivery modes, base-stock policies.
1. Introduction

In this note, we construct a counterexample to establish that a base-stock policy is not optimal in general for inventory models with multiple delivery modes. In most of the studies on inventory models with replenishment leadtime options, it is assumed that there are two consecutive procurement modes available (e.g., Neuts 1964, Lawson et al. 2000, Sethi et al. 2001, Muharremoglu et. al. 2003). That is, the leadtimes of the two modes vary by exactly one period. This assumption leads to the optimality of a base-stock policy. Whittemore et al. (1997) formulate a problem for general two delivery modes and derive explicit formulas for the optimal ordering quantities when the modes are consecutive. They also comment that the optimal policy for two nonconsecutive modes may not have a simple structure. In the case of three consecutive delivery modes, Fukuda (1964) first considers a special case when orders are placed only every other period, in which case a base-stock policy is shown to be optimal. Zhang (1996) extends Fukuda’s model by allowing three consecutive modes ordered every period, and claims the optimality of a base-stock policy under certain conditions. Her claim is untrue as evident from the counterexample presented in this note. Feng et al. (2005) show that there exist optimal base-stock levels for the two faster modes in an inventory model with three consecutive delivery modes and demand forecast updates.

We formulate a general problem in Section 2. We analyze the policy structure through an example in Section 3, and discuss the separability of the cost functions in Section 4. We conclude our study in Section 5.

2. Problem Formulation

We consider a finite horizon periodic review inventory system with \( N \) consecutive delivery modes. In the case of non-consecutive delivery modes, one can insert fictitious delivery modes as suggested in Sethi et al. (2001) to transform the problem into one with consecutive modes by setting the cost for any fictitious mode to equal that for the next faster real mode. Setting costs this way implies that we only consider policies that do not issue orders using fictitious modes.

An order \( Q^i_\ell \) in period \( \ell \) via the \( i^{th} \) delivery mode, \( i = 1, 2, ..., N \), is an order associated with a leadtime of \( i \) periods, and is termed the type \( i \) order. We denote \( q^i_\ell \) as the realized
value of $Q_i^\ell$. A reference (pre-order) inventory position for an order $Q_i^\ell$ is defined as the sum of on-hand inventory in period $\ell$ plus all previous and current orders placed before $Q_i^\ell$ and to be delivered by the time $Q_i^\ell$ is delivered, i.e., at the end of period $\ell + i - 1$. A post-order inventory position for an order $Q_i^\ell$ is defined as the sum of its reference inventory position and $Q_i^\ell$. The notation is given below and the inventory positions are presented in Table 1.

$Q_k^i = \text{the amount of type } i \text{ order that is placed at the beginning of period } k \text{ and will arrive at the end of period } k + i - 1, 1 \leq k \leq T, 1 \leq i \leq N;$

c_k^i = \text{the unit procurement cost of type } i \text{ order in period } k, 1 \leq k \leq T, 1 \leq i \leq N;$

$D_k = \text{the demand in period } k \text{ (materialized after delivery of orders), } 1 \leq k \leq T;$

$x_k = \text{the inventory/backlog level at the beginning of period } k, 1 \leq k \leq T;$

$p_k^i = \sum_{j=i+1}^N q_{k+i-j}^j, \text{ the amount of in-transit orders at the beginning of period } k \text{ that will arrive at the end of period } k + i - 1, 1 \leq k \leq T, 1 \leq i \leq N;$

$y_k = x_k + p_k^1, \text{ the reference inventory position for } Q_k^1 \text{ at the beginning of period } k, 1 \leq k \leq T;$

$z_k^i = \text{the post-order inventory position for } Q_k^i \text{ viewed at the beginning of period } k, 1 \leq k \leq T, 1 \leq i \leq N;$

$H_k(\cdot) = \text{the inventory holding/backlog cost in period } k \text{ assessed on the beginning inventory } x_k, 1 \leq k \leq T;$

$H_{T+1}(\cdot) = \text{the costs for the ending inventory/backlog.}$

We assume that that $E[D_k] < \infty$ for each $k$. Furthermore, the inventory cost functions $H_k(x)$ is nonnegative convex with $H_k(0) = 0$ for each $k$. The objective is to choose a sequence of orders $\{Q_k^1, \ldots, Q_k^N\}_{1 \leq k \leq T}$ so as to minimize the total expected cost given by

$$J_1(x_1, p_1^1, \ldots, p_1^{N-1}; \{Q_k^1, \ldots, Q_k^N\}_{1 \leq k \leq T}) = H_1(x_1) + E \left\{ \sum_{k=1}^T \left[ \sum_{j=1}^N c_k^j Q_k^j + H_{k+1}(X_{k+1}) \right] \right\}, \quad (1)$$

subject to the inventory dynamics

$$X_{k+1} = x_k + p_k^1 + Q_k^1 - D_k, \quad 1 \leq k \leq T. \quad (2)$$

Let $W_k(y_k, p_k^2, \ldots, p_k^{N-1})$ denote the cost-to-go function in period $k$. Then the function satisfies
One can show, along the lines of the proof of Theorem 4.3 in Sethi et al. (2001), that there exist functions $z^*_k(y_k, p^2_k, ..., p^N_k)$, $1 \leq k \leq T - 1$, which minimize the right-hand side in (3).

3. An Example

In this section, we examine a three period example with three consecutive modes. There are three types of orders, namely, fast $Q^1$, medium $Q^2$, and slow $Q^3$, with stationary ordering costs $c^1 = 3$, $c^2 = 2$, and $c^3 = 1$, respectively. The demand $D_1$ for the first period follows a uniform distribution over $[0, 20]$, and the demands for the last two periods are deterministic $D_2 = D_3 = 20$. Holding and backlog costs at the end of each period are given by

$$H_4(x) = \begin{cases} 0 & \text{if } x \geq 0, \\ -10x & \text{if } x < 0; \end{cases} \quad H_3(x) = \begin{cases} 3x & \text{if } x \geq 0, \\ -4x & \text{if } x < 0; \end{cases} \quad H_2(x) = \begin{cases} 2x & \text{if } x \geq 0, \\ -4x & \text{if } x < 0. \end{cases}$$

3.1 The Optimal Ordering Policies

The optimal cost function for the third period is given by

$$W_3(y_3) = \begin{cases} 0 & \text{if } y_3 \geq 20, \\ 60 - 3y_3 & \text{if } y_3 < 20. \end{cases}$$
The optimal base-stock level is $z_3^1 = 20$.

The optimal cost function for period 2 is given by

$$W_2(y_2, q_1^3) = \begin{cases} 
100 - 2q_1^3 - 3y_2 & \text{if } q_1^3 \leq 20, y_2 \leq 20, \\
20 - 2q_1^3 + y_2 & \text{if } 20 < y_2 \leq 40 - q_1^3, \\
60 - 3y_2 & \text{if } q_1^3 > 20, y_2 \leq 20, \\
-60 + 3y_2 & \text{if } y_2 > \max\{20, 40 - q_1^3\}\end{cases}$$

Here, $p_2^2 = q_1^3$. The optimal base-stock levels are $(z_2^1, z_2^2) = (20, 40)$.

The optimal cost function for period 1 is given by

$$W_1(y_1, q_0^3) = \begin{cases} 
\frac{388}{3} - 2q_0^3 - 3y_1 & \text{if } y_1 \leq 0, q_0^3 \leq \frac{49}{3}, \\
\frac{388}{3} - 2q_0^3 - 6y_1 + \frac{3}{20}(y_1)^2 & \text{if } 10 < y_1 \leq \min\{20, \frac{70}{3} - q_0^3\}, \\
\frac{388}{3} - 2q_0^3 & \text{if } 20 < y_1 \leq \frac{70}{3} - q_0^3, \\
\frac{226}{3} - 9q_0^3 + \frac{3}{20}(q_0^3)^2 - 7y_1 + \frac{3}{10}q_0^3y_1 + \frac{3}{20}(y_1)^2 & \text{if } \max\{20, \frac{70}{3} - q_0^3\} < y_1 \leq 30 - q_0^3, \\
150 - 6q_0^3 + \frac{3}{10}(q_0^3)^2 - 4y_1 + \frac{3}{10}q_0^3y_1 + \frac{3}{20}(y_1)^2 & \text{if } \max\{20, 30 - q_0^3\} < y_1 \leq 40 - q_0^3, \\
-55 + 2q_0^3 + 4y_1 & \text{if } \max\{20, 40 - q_0^3\} < y_1 \leq 50 - q_0^3, \\
70 - 3q_0^3 + \frac{3}{10}(q_0^3)^2 - y_1 + \frac{3}{10}q_0^3y_1 + \frac{3}{20}(y_1)^2 & \text{if } \max\{20, 50 - q_0^3\} < y_1 \leq 60 - q_0^3, \\
-110 + 3q_0^3 + 5y_1 & \text{if } y_1 > \max\{20, 60 - q_0^3\}, \\
\frac{388}{3} - 4q_0^3 + \frac{3}{10}(q_0^3)^2 - 3y_1 & \text{if } y_1 \leq \frac{50}{3} - \frac{7}{3}q_0^3, 40 - q_0^3 < q_0^3 \leq \frac{80}{3}, \\
210 - 9q_0^3 + \frac{3}{10}(q_0^3)^2 - 13y_1 + \frac{3}{10}q_0^3y_1 + \frac{3}{10}(y_1)^2 & \text{if } \max\{\frac{3}{10} - \frac{7}{3}q_0^3, \frac{3}{10} - q_0^3\} < y_1 \leq \min\{20, 30 - q_0^3\}, \\
165 + 6q_0^3 + \frac{1}{5}q_0^3 - 10y_1 + \frac{1}{5}q_0^3y_1 + \frac{1}{5}(y_1)^2 & \text{if } \max\{14 - \frac{2}{5}q_0^3, 30 - q_0^3\} < y_1 \leq \min\{20, 40 - q_0^3\}, \\
5 + 2q_0^3 - 2y_1 + \frac{3}{20}(y_1)^2 & \text{if } \max\{0, 40 - q_0^3\} < y_1 \leq \min\{20, 50 - q_0^3\}, \\
130 - 3q_0^3 + \frac{1}{10}(q_0^3)^2 - 7y_1 + \frac{1}{10}q_0^3y_1 + \frac{1}{10}(y_1)^2 & \text{if } \max\{0, 50 - q_0^3\} < y_1 \leq \min\{20, 60 - q_0^3\}, \\
-50 + 3q_0^3 - y_1 + \frac{3}{20}(y_1)^2 & \text{if } y_1 > \max\{0, 60 - q_0^3\}, \\
\frac{26}{3} + q_0^3 - 3y_1 & \text{if } y_1 \leq 35 - q_0^3, q_0^3 > 35, \\
116 - \frac{1}{5}q_0^3 + \frac{1}{10}(q_0^3)^2 - 3y_1 & \text{if } y_1 \leq 14 - \frac{2}{5}q_0^3, \frac{26}{3} < q_0^3 \leq 35, \\
165 - 6q_0^3 + \frac{1}{10}(q_0^3)^2 - 10y_1 + \frac{1}{10}q_0^3y_1 + \frac{1}{10}(y_1)^2 & \text{if } 35 - q_0^3 < y_1 \leq \min\{0, 30 - q_0^3\}, \\
5 + 2q_0^3 - 2y_1 & \text{if } 30 - q_0^3 < y_1 \leq \min\{0, 40 - q_0^3\}, \\
130 - 3q_0^3 + \frac{1}{10}(q_0^3)^2 - 7y_1 + \frac{1}{10}q_0^3y_1 + \frac{1}{10}(y_1)^2 & \text{if } 50 - q_0^3 < y_1 \leq \min\{0, 60 - q_0^3\}, \\
-50 + 3q_0^3 - y_1 & \text{if } y_1 \leq \min\{0, 60 - q_0^3\}.
\end{cases}$$

In Fig. 1., we depict the various regions involved in defining $W_1(y_1, q_0^3)$ in (4). Here, $p_1^2 = q_0^3$.

There are optimal base-stock levels for the first two modes, which are given by

$$\begin{align*}
(z_1^1, z_1^2) = \begin{cases} 
(35 - q_0^3, 35) & \text{if } q_0^3 \geq 35, \\
(14 - \frac{2}{5}q_0^3, 14 + 3q_0^3) & \text{if } \frac{40}{3} \leq q_0^3 < 35, \\
(50 - \frac{7}{3}q_0^3, 50 + 3q_0^3) & \text{if } \frac{30}{3} \leq q_0^3 < \frac{80}{3}, \\
(10, \frac{70}{3}) & \text{if } 0 \leq q_0^3 < \frac{40}{3}.
\end{cases}
\end{align*}$$

### 3.2 Discussion

We focus on analyzing the optimal decision for $Q_1^{3*}$. Fig. 2 shows how $Q_1^{3*}$ changes for given the reference inventory position $z_1^2$, which does not follow a base-stock structure (This also
gives a counterexample to Lemma 1 in Zhang 1996). To further explore the reason, we consider the optimal post-order inventory positions in period 1:

\[
(z_1^*, z_2^*, z_3^*) = \begin{cases}
(10, \frac{70}{3}, \frac{130}{3}) & \text{if } y_1 \leq 10, \\
(y_1, y_1 + q_0^3, y_1 + q_0^3 + 20) & \text{if } 10 < y_1 \leq \frac{70}{3} - q_0^3, \\
(y_1, y_1 + q_0^3, 50) & \text{if } \frac{70}{3} - q_0^3 < y_1 \leq 30 - q_0^3, \\
(y_1, y_1 + q_0^3, y_1 + q_0^3) & \text{if } 30 - q_0^3 < y_1 \leq 50 - q_0^3, \\
(\frac{50}{3} - \frac{1}{2} q_0^3, \frac{50}{3} + q_0^3, \frac{110}{3} + q_0^3) & \text{if } y_1 \leq \frac{50}{3} - \frac{1}{2} q_0^3, \\
(y_1, y_1 + q_0^3, y_1 + q_0^3 + 20) & \text{if } \frac{50}{3} - \frac{1}{2} q_0^3 < y_1 \leq 30 - q_0^3, \\
(y_1, y_1 + q_0^3, 50) & \text{if } 30 - q_0^3 < y_1 \leq 50 - q_0^3, \\
(y_1, y_1 + q_0^3, y_1 + q_0^3) & \text{if } y_1 \geq 50 - q_0^3, \\
(14 - \frac{2}{3} q_0^3, \frac{50}{3} + q_0^3, 50) & \text{if } y_1 \leq 14 - \frac{2}{3} q_0^3, \\
(y_1, y_1 + q_0^3, 50) & \text{if } 14 - \frac{2}{3} q_0^3 < y_1 \leq 50 - q_0^3, \\
(y_1, y_1 + q_0^3, y_1 + q_0^3) & \text{if } y_1 \geq 50 - q_0^3, \\
(35 - q_0^3, 35, 50) & \text{if } y_1 \leq 35 - q_0^3, \\
(y_1, y_1 + q_0^3, 50) & \text{if } 35 - q_0^3 < y_1 \leq 50 - q_0^3, \\
(y_1, y_1 + q_0^3, y_1 + q_0^3) & \text{if } y_1 \geq 50 - q_0^3, \\
q_0^3 & \text{if } q_0^3 \geq 35;
\end{cases}
\]

Figure 1: The cost function for period 1.

Figure 2: The function $Q_{z_1^*}(z_1)$.
It seems that the ideal order-up-to level for \( Q_1^3 \) would be 50, if the ordering policy for the slow mode were to follow a base-stock policy. However, this cannot always be achieved when \( q_0^3 \) is low (\( q_0^3 \leq 80/3 \)).

Table 2 illustrates the optimal ordering decisions when \( 0 \leq q_0^3 < 40/3 \). If \( y_1 < 70/3 - q_0^3 \), we have to order a positive \( Q_2^{1*} \) in period 2 and 20 units of \( Q_1^{3*} \). However, the inventory position \( z_1^{3*} \) does not account for the value of \( Q_1^{3*} \). Thus, the optimal decision is not to bring \( z_1^{3*} \) up to 50. When \( y_1 \geq 70/3 - q_0^3 \), there is no fast order placed in period 2. As a result, the optimal inventory position \( z_1^{3*} \) should be as close to the target level 50 as possible. One may think that if we modify the reference inventory position for \( Q_1^3 \) to include the anticipated order \( Q_1^{3_2} \), then we could restore the base-stock policy. However, this would not work in the case of general stochastic demand.

<table>
<thead>
<tr>
<th>Period 1</th>
<th>( y_1 )</th>
<th>((-\infty, 10])</th>
<th>((10, \frac{40-q_0^3}{10}])</th>
<th>((\frac{40-q_0^3}{10}, 30-q_0^3])</th>
<th>((30-q_0^3, 40-q_0^3])</th>
<th>((40-q_0^3, 50-q_0^3])</th>
<th>((50-q_0^3, \infty])</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z_1^{1*} + q_0^3 )</td>
<td>10 + ( q_0^3 )</td>
<td>10 + ( q_0^3 )</td>
<td>( y_1 + q_0^3 )</td>
<td>( y_1 + q_0^3 )</td>
<td>( y_1 + q_0^3 )</td>
<td>( y_1 + q_0^3 )</td>
<td>( y_1 + q_0^3 )</td>
</tr>
<tr>
<td>( z_1^{2*} )</td>
<td>( \frac{40 - q_0^3}{3} )</td>
<td>( \frac{40 - q_0^3}{3} )</td>
<td>( y_1 + q_0^3 )</td>
<td>( y_1 + q_0^3 )</td>
<td>( y_1 + q_0^3 )</td>
<td>( y_1 + q_0^3 )</td>
<td>( y_1 + q_0^3 )</td>
</tr>
<tr>
<td>( z_1^3 )</td>
<td>20</td>
<td>20</td>
<td>( y_1 + q_0^3 + 20 )</td>
<td>( 50 - y_1 - q_0^3 )</td>
<td>( 50 - y_1 - q_0^3 )</td>
<td>( y_1 + q_0^3 )</td>
<td>( y_1 + q_0^3 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Period 2</th>
<th>( y_2 )</th>
<th>((-\infty, d_1])</th>
<th>((d_1, d_2])</th>
<th>((d_2, +\infty])</th>
<th>((d_1 - y_2^2, 0])</th>
<th>((0, \infty])</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z_2^{1*} )</td>
<td>( \frac{40-y_2^2}{40} )</td>
<td>( \frac{40-y_2^2}{40} )</td>
<td>( \frac{40-y_2^2}{40} )</td>
<td>( \frac{40-y_2^2}{40} )</td>
<td>( \frac{40-y_2^2}{40} )</td>
<td>( \frac{40-y_2^2}{40} )</td>
</tr>
<tr>
<td>( z_2^{2*} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \max(60 - y_1 - q_0^3, 0) )</td>
</tr>
</tbody>
</table>

When \( q_0^3 \) is lower than 80/3, there is a positive probability that \( Q_2^{1*} > 0 \). Since the reference inventory position for the type 3 order in period 1 does not take this order into account, the optimal ordering policy in period 1 is not a base-stock policy. When \( q_0^3 \) is greater than 80/3, we always have \( Q_1^{1*} = 0 \) and the optimal policy in period 1 follows a base-stock policy.

We also observe from our example that when the type 3 orders do not follow base-stock policy, the order quantity \( Q_1^{3*} \) is constant 20. However, this observation cannot be generalized. If we modify the holding/backlog cost in period 3 to be \( H_3(x) = 0.5x^2 \), then \( Q_1^{3*} \) does not have this property any longer. The corresponding \( Q_1^{3*}(z_1^2) \) is given in Fig. 3.
Figure 3: $Q^*_{3}(z^2_1)$ for nonlinear holding costs.

4. Separability and the Base-Stock Policies

The example in the last section shows that the optimal ordering policies for inventory models with more than two modes are no longer base-stock policies in general. Next, we examine the relationship between an optimal base-stock policy and the separability of the cost functions. The next result establishes the existence of optimal base-stock levels for the first two modes in general inventory systems with multiple delivery modes.

**Proposition 1** The minimand in the right-hand side of (3) can be expressed as $G_1(z^1_k) + G_2(z^2_k, ..., z^N_k)$, where $G_1(z^1_k)$ is convex in $z^1_k$ and $G_2(z^2_k, ..., z^N_k)$ is jointly convex in $(z^2_k, ..., z^N_k)$. Let $(z^{1*}_k, ..., z^{N*}_k)$ be the minimizing vector in (3). Then there exist two base stocks $\bar{z}^1_k$ and $\bar{z}^2_k$ such that

$$z^{1*}_k = \bar{z}^1_k \lor y_k, \quad z^{2*}_k = \bar{z}^2_k \lor (z^{1*}_k + p^2_k).$$

In general, the cost-to-go function defined in (3) is not separable in reference inventory positions $(z^2_k, ..., z^N_k)$. If it were, it can be easily shown that there would exist optimal base-stock levels in period $k$ for type 3 and higher orders. This may happen in some very special cases. One such example is Fukuda’s (1964) model of three consecutive modes where orders are placed in every other period. There the resulting cost-to-go function is separable in the reference inventory positions, and the optimal ordering policy is a base-stock policy. Furthermore, the following proposition represents a generalization of Fukuda’s (1964) result.

**Proposition 2** If the first mode in period $k + 1$ is not used, then the first three modes in period $k$ have optimal base-stock levels.
Remark 1 Note that the proof cannot be generalized for more than three modes. That is, if the first two modes in period $k + 1$ are not used, then the fourth mode in period $k$ may not have an optimal base stock.

5. Concluding Remarks

We have shown that the optimality of a base-stock policy is closely related to the structure of the cost-to-go function. Since the cost-to-go function for each period is separable in the post-order inventory positions for the first two modes, there exist optimal base-stock levels for the first two modes. In general, the cost-to-go function for a given period is not separable in post-order inventory positions of higher modes. Thus, the optimal ordering decision for type 3 or higher order may not have a base-stock structure. Our discussion shows that the base-stock policy fails to be optimal even under very restrictive conditions, e.g., uniform demand, linear holding costs. The intuitive reason is that the optimal post-order inventory position for a mode of type 3 or higher cannot, in general, anticipate the future order quantities. Moreover, the dependence of the optimal order quantities on the reference inventory position and in-transit orders may be quite complex.

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—, —, —, —, —. 2004. Optimality and Nonoptimality of Base-stock Policy in Inventory Model with Multiple Delivery Modes. (the unabridged version of this paper). University of Texas at Dallas, Richardson, TX.


The proof of Proposition 1 uses the following lemma.

**Lemma 1** Suppose $G(x^1, \ldots, x^n)$ is jointly convex in $(x^1, \ldots, x^n)$. Define $G^o(x^1, \ldots, x^k) = \min \{G(x^1, \ldots, x^n) | x^j \geq x^{j-1} + a_j \text{ for } j = k+1, \ldots, n\}$ for $k = 1, \ldots, N$. Then $G^o(x^1, \ldots, x^k)$ is jointly convex in $(x^1, \ldots, x^k)$.

**Proof.** We show that $G^o(x^1, \ldots, x^k)$ is jointly convex by induction on $k$. Clearly, when $k = N$, $G^o(x^1, \ldots, x^N) = G(x^1, \ldots, x^N)$ is jointly convex. Suppose that $G^o(x^1, \ldots, x^{k+1})$ is jointly convex in $(x^1, \ldots, x^{k+1})$. Then,

$$G^o(x^1, \ldots, x^k) = \min \{G(x^1, \ldots, x^N) | x^j \geq x^{j-1} + a_j \text{ for } j = k+1, \ldots, N\}$$

$$= \min \{G^o(x^1, \ldots, x^{k+1}) | x^{k+1} \geq x^k + a_{k+1}\}.$$

Now define $\tilde{x}^{k+1}(x^1, \ldots, x^k) = \arg \min_{x^{k+1}} G^o(x^1, \ldots, x^{k+1})$, then

$$G^o(x^1, \ldots, x^k) = G^o(x^1, \ldots, \tilde{x}^{k+1}(x^1, \ldots, x^k) \vee (x^k + a_{k+1})).$$

Clearly, $G^o\left(x^1, \ldots, \tilde{x}^{k+1}(x^1, \ldots, x^k)\right)$ is convex since the lower envelope of a convex function is convex. Thus, $G^o(x^1, \ldots, x^k)$ is convex.

**Proof of Proposition 1.** The minimand in the right-hand side of (3) is easily seen to be separable in $z^1_k$ and $(z^2_k, \ldots, z^N_k)$. By Lemma 1, we have

$$W_k(y_k, p^2_k, \ldots, p^{N-1}_k)$$
Thus, for any 

\[
T \text{ o see the last inequality, we note that the right-hand side is convex in } z_k^1. \text{ We consider two cases.}
\]

\textbf{Case 1}: \( \hat{z}_k^1 + p_k^2 \leq \bar{z}_k^2 \). First note that in this case \( \hat{z}_k^1 \leq \hat{z}_k^1, \bar{z}_k^1 = \hat{z}_k^1 \). We have \( \hat{z}_k^1 + p_k^1 \leq \bar{z}_k^2 \). Thus, for any \( z_k^1 \geq y_k \),

\[
G_1(\hat{z}_k^1 \lor y_k) + G_2(\bar{z}_k^2 \lor (y_k \lor \hat{z}_k^1 + p_k^2)) \leq G_1(z_k^1 \lor y_k) + G_2(\bar{z}_k^2 \lor (y_k \lor z_k^1 + p_k^2)).
\]

To see the last inequality, we note that the right-hand side is convex in \( z_k^1 \). If \( y_k \leq \hat{z}_k^1 \), the right-hand side is minimized at \( z_k^1 = \hat{z}_k^1 = \hat{z}_k^1 \). If \( y_k > \hat{z}_k^1 \), the right-hand side is minimized at \( z_k^1 = y_k \). Hence, the minimizer is \( z_k^{1*} = \bar{z}_k^1 \lor y_k = \bar{z}_k^1 \lor y_k \) and \( z_k^{2*} = \bar{z}_k^2 \lor (z_k^{1*} + p_k^2) \).

\textbf{Case 2}: \( \hat{z}_k^1 + p_k^2 > \bar{z}_k^2 \). In this case the minimizer must be such that \( z_k^{1*} + p_k^2 = z_k^{2*} \). Note that \( \bar{z}_k^1 < \bar{z}_k^2 \) and \( \hat{z}_k^1 < \hat{z}_k^1 + p_k^2 \). So \( \bar{z}_k^1 = \bar{z}_k^1 \) and \( \hat{z}_k^1 = \hat{z}_k^1 < \hat{z}_k^1 + p_k^2 \). Thus, for any \( z_k^1 \geq y_k \),

\[
G_1(\bar{z}_k^1 \lor y_k) + G_2(\hat{z}_k^1 \lor y_k + p_k^2) = G_1(\hat{z}_k^1 \lor y_k) + G_2(\bar{z}_k^2 \lor (\bar{z}_k^1 \lor y_k + p_k^2))
\leq G_1(z_k^1 \lor y_k) + G_2(\bar{z}_k^2 \lor (z_k^1 \lor p_k^2)).
\]

Similar to Case 1, if \( y_k \leq \hat{z}_k^1 \), then the right-hand side is minimized at \( \hat{z}_k^1 \). Otherwise, it is minimized at \( y_k \). Hence, the minimizer is \( z_k^{1*} = \hat{z}_k^1 \lor y_k = \bar{z}_k^1 \lor y_k \) and \( z_k^{2*} = \bar{z}_k^2 \lor (z_k^{1*} + p_k^2) \).

\textbf{Proof of Proposition 2}. The cost function for period \( k \) can be written as

\[
J_k(z_k^1, \ldots, z_k^N) = c_k^1(z_k^1 - y_k) + \sum_{j=2}^{N} c_k^j(z_k^j - z_k^{j-1} - p_k^j) + EH_{k+1}(z_k^1 - D_k)
\]

\[
+ EW_{k+1}(z_k^2 - D_k, z_k^3 - z_k^2, \ldots, z_k^N - z_k^{N-1})
\]

\[
= c_k^1(z_k^1 - y_k) + \sum_{j=2}^{N} c_k^j(z_k^j - z_k^{j-1} - p_k^j) + EH_{k+1}(z_k^1 - D_k) +
\]

\[
= \min\{G_1(z_k^1) + G_2(z_k^1)|z_k^1 \geq y_k, z_k^1 \geq \hat{z}_k^1 + p_k^1, j = 2, \ldots, N\}
\]

\[
= \min\{G_1(z_k^1) + G_2(z_k^2)|z_k^1 \geq y_k, z_k^2 \geq \hat{z}_k^1 + p_k^1\}.
\]
\[ +E \min_{z_{k+1}^3 \geq z_k^2, \ z_{k+1}^3 \geq z_{k+1}^2 + z_k^2} \left\{ c_{k+1}^3 (z_{k+1}^3 - z_k^4 + D_k) + \sum_{j=4}^N c_j^j (z_{k+1}^j - z_{k+1}^{j-1} - z_{k+1}^{j-1} + z_k^j) \right\} \]

Clearly, the last expression is separable in \((z_k^1, z_k^2, z_k^3)\). Hence, the result follows from Proposition 5 of Feng et al. (2004).

\[ \square \]