Example:

Consider the periodic square wave $x[t]$ shown in the figure
- determine the complex fourier series of $x[t]$

(a) let

$$x[t] = \sum_{k = -\infty}^{\infty} a_k e^{jk\omega_0 t}$$

we have

$$a_k = \frac{1}{T_0} \int_{T_0 - T_0/2}^{T_0/2} x[t] e^{-jk\omega_0 t} dt = \frac{1}{T_0} \int_{T_0 - T_0/4}^{T_0/4} A e^{-jk\omega_0 t} dt$$

$$\omega_0 = \frac{2\pi}{T_0}$$
Example contd:

\[
\begin{align*}
  &= \frac{A}{-jk\omega T_0} \left( e^{-jk\omega T_0/4} - e^{jk\omega T_0/4} \right) \\
  &= \frac{A}{-jk2\pi} \left( e^{-jk\pi/2} - e^{jk\pi/2} \right) = \frac{A}{2\pi} \sin\left( \frac{k\pi}{2} \right)
\end{align*}
\]

Thus

\[
\begin{align*}
  a_k &= 0 & k &= 2m \neq 0 \\
  a_k &= (-1)^m \frac{A}{k\pi} & k &= 2m+1 \\
  a_0 &= \frac{1}{T} \int_0^T x(t) dt = \frac{T}{0} A dt = A/2
\end{align*}
\]
Example contd:

Hence
\[ a_0 = \frac{A}{2}, \quad a_{2m} = 0, m \neq 0, \quad a_{2m+1} = (-1)^m \frac{A}{(2m+1)\pi} \]

and we obtain
\[ x(t) = \frac{A}{2} + \frac{A}{\pi} \sum_{m=\infty}^{\infty} e^{j(2m+1)\omega_0 t} \]

Expressing in terms of the cosines
\[ \frac{c_0}{2} = c_0 = \frac{A}{2}, \quad c_{2m} = 2 \text{Re}[a_{2m}] = 0, m \neq 0 \]
\[ c_{2m+1} = 2 \text{Re}[a_{2m+1}] = (-1)^m \frac{A}{(2m+1)\pi} \]
\[ b_k = -2 \text{Im}[a_k] = 0 \]
Example contd:

Substituting these values into the series, we obtain

\[ x[t] = \frac{A}{2} + \frac{2A}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \cos((2m+1)\omega_0 t) \]

\[ = \frac{A}{2} + \frac{2A}{\pi} \left( \cos\omega_0 t - \frac{1}{3} \cos3\omega_0 t + \frac{1}{5} \cos5\omega_0 t + \ldots \right) \]

Note that \( x[t] \) is even; thus, \( x[t] \) contains only a d.c. term and cosine terms.
Convergence of the Fourier Series

Examine a problem of approximating a given periodic signal \( x[t] \) by a linear combination of a finite number of harmonically related complex exponentials - that is by a finite series of the form

\[
x_N[t] = \sum_{k=-N}^{N} a_k e^{j\omega_0 t}
\]

let \( e_N(t) \) denote the approximation error.

\[
e_N(t) = x[t] - x_N[t] = x[t] - \sum_{k=-N}^{N} a_k e^{j\omega_0 t}
\]

Definition:

Energy of a periodic signal in one period is given by:

\[
E = \int_T |v(t)|^2 \, dt
\]

\[
E_N = \int_T |e_N(t)|^2 \, dt
\]
Convergence of the Fourier Series

What are the conditions that the energy $E_N$ in the approximation error approaches zero as we add more and more terms as $N$ approaches infinity. That is

$$e[t] = x[t] - \sum_{k=-\infty}^{\infty} a_k e^{j k \omega_0 t}$$

$$\int_{T} ||e[t]||^2 dt = 0$$

As new terms are now added $E_N$ decreases. Fortunately there are no convergence difficulties for large classes of periodic signals.

It is known that a periodic signal $x[t]$ has a fourier series representation if it satisfies the following Dirichlet conditions:

1. $x[t]$ is absolutely integrable over any period, that is

$$\int_{t_0}^{t_0+T} |x[t]| dt < \infty$$

2. $x[t]$ has a finite number of maxima and minima over any period

3. $x[t]$ has only a finite number of discontinuities over any period
Parseval's Theorem

$x[t]$ is periodic with period of $T$. The average power $P$ in one period of the periodic signal is defined by

$$P = \frac{1}{T} \int_{T/2}^{T/2} |x[t]|^2 dt$$

By Parseval's Theorem, the average power is given by

$$P = \sum_{k=-\infty}^{\infty} |a_k|^2$$

average power is the energy transferred per unit time.
Properties of Continuous-Time Fourier Series

Use a shorthand notation to indicate the relationship between a periodic signal and its Fourier series coefficients.

\[ x[t] \rightarrow T \rightarrow \text{Period} \]

\[ \omega_0 = \frac{2\pi}{T} \rightarrow \text{fundamental frequency} \]

\[ x[t]^{-F.S.} \rightarrow a_k \]

This notation signifies the pairing of a periodic signal with its Fourier series coefficients.
Linearity

Suppose that \( x[t] \) and \( y[t] \) are two periodic signals with period \( T \), then

\[
x[t] \xrightarrow{F.S.} a_k \\
y[t] \xrightarrow{F.S.} b_k
\]

Any linear combination of \( x[t] \) and \( y[t] \) will be periodic with a period of \( T \)

\[
z[t] = Ax[t] + By[t]
\]

\[
z[t] \xrightarrow{F.S.} c_k = Aa_k + Bb_k
\]
Time Shifting

The period of the time shifted signal is preserved.

\[ x[t] \overset{F.S.}{\rightarrow} a_k \]

\[ y[t] = x[t - t_0] \overset{F.S.}{\rightarrow} b_k = a_k e^{-jk(2\pi / T)t_0} \]

\[ b_k = \frac{1}{T} \int_{-T/2}^{T/2} x[t - t_0] e^{-jk\omega_0 t} dt = a_k e^{-jk\omega_0 t_0} \]