Problem 1.3.5

The sample space of the experiment is:
\[ S = \{ LF, RF, LW, BW \} \]

From the problem statement, we know that \( P(LF) = 0.5 \), \( P(RF) = 0.2 \)
and \( P(BW) = 0.2 \). This implies \( P(LW) = 1 - 0.5 - 0.2 - 0.2 = 0.1 \).
These questions can be answered using Theorem 1.6.

(a) The probability that a program is slow is
\[
P(LW) = P(LW) + P(BW) = 0.1 + 0.2 = 0.3
\]

(b) The probability that a program is big is
\[
P(B) = P(RF) + P(BW) = 0.2 + 0.2 = 0.4
\]

(c) The probability that a program is slow or big is
\[
P(LW \cup B) = P(LW) + P(B) - P(LW) = 0.3 + 0.4 - 0.2 = 0.5
\]

Problem 1.3.6

The sample space is:
\[ S = \{ HF, HW, MF, MW \} \]

The problem statement tells us that \( P(HF) = 0.2 \), \( P(MW) = 0.1 \)
and \( P(F) = 0.5 \). We can use these facts to find
the probabilities of other outcomes. In particular,
\[
P(F) = P(HF) + P(MF)
\]
This implies
\[ P[M|F] = P[F] - P[H|F] = 0.5 - 0.2 = 0.3 \]
Also, since the probabilities must sum to 1,
\[ = 1 - 0.2 - 0.3 - 0.1 = 0.4 \]
Now, that we have found the probabilities of the first corner, finding any other probability is easy.
(a) The probability a cell phone is slow is
\[ P[W] = P[H|W] + P[M|W] = 0.4 + 0.1 = 0.5 \]
(b) The probability that a cell phone is mobile and fast is
\[ P[M|F] = 0.3 \]
(c) The probability that a cell phone is handheld is,
\[ P[H] = P[H|F] + P[H|W] \]
\[ = 0.2 + 0.4 = 0.6 \]
Problem 1.5.2

(a) From the given probability distribution of billed minutes $M$, the probability that a call is billed for more than 3 minutes is,

\[ P[L] = 1 - P[3 \text{ or fewer billed minutes}] \]


\[ = 1 - \alpha - \alpha(1-\alpha) - \alpha(1-\alpha)^2 \]

\[ = (1-\alpha)^3 = 0.57 \]

(b) The probability that a call will be billed for 9 minutes or less is,

\[ P[9 \text{ minutes or less}] = \sum_{n=0}^{9} \alpha(1-\alpha)^n = 1 - (0.57)^5 \]

Problem 1.4.2

Let $S_i$ denote the outcome that the roll is $i$, so for $i = 1\ldots6$, $R_i = \{S_i\}$. Similarly, $G_{i,j} = \{S_{i,j}, \ldots, S_{6,j}\}$.

(a) Since $G_{i,j} = \{S_i, S_3, S_5, S_7, \ldots, S_{6,j}\}$ and all outcomes have probability \(\frac{1}{6}\),

\[ P[G_{i,j}] = \frac{1}{6} \].

The event $R_2G_{i,j} = \{S_2, S_3\}$ and $P[R_2G_{i,j}] = \frac{1}{6}$

so that

\[ P[R_2/G_{i,j}] = \frac{P[R_2G_{i,j}]}{P[G_{i,j}]} = \frac{1}{5}. \]
(b) The conditional probability that 6 is rolled given that the roll is greater than 3 is,
\[
P[R_6/G_3] = \frac{P[R_6 \cap G_3]}{P[G_3]} = \frac{P[R_6]}{P[G_3]} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}.
\]

(c) The event $E$ that the roll is even is $E = \{s_2, s_4, s_6\}$ and has probability $2/3$. The joint probability of $G_3$ and $E$ is
\[
P[G_3 \cap E] = P[s_4, s_6] = \frac{1}{2}.
\]
The conditional probabilities of $G_3$ given $E$ are,
\[
P[G_3/E] = \frac{P[G_3 \cap E]}{P[E]} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}.
\]

Problem 1.4.3

Since the 2 of clubs is an even numbered card, $E = C_2 \cap E$
so that $P[C_2 \cap E] = P[C_2] = \frac{1}{3}$. Since $P[E] = \frac{2}{3}$,
\[
P[C_2/E] = \frac{P[C_2 \cap E]}{P[E]} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}.
\]
The probability that an even numbered card is picked given that the 2 is picked is
\[
P[E/C_2] = \frac{P[C_2 \cap E]}{P[C_2]} = \frac{\frac{1}{3}}{\frac{1}{3}} = 1.
\]
Problem 1.6.7

(a) Since \( A \cap B = \emptyset \), \( P[A \cap B] = 0 \). To find \( P[B] \), we can write
\[
P[A \cup B] = P[A] + P[B] - P[A \cap B]
\]
\[
\frac{5}{8} = \frac{3}{8} + P[B] - 0
\]
Thus, \( P[B] = \frac{1}{4} \). Since \( A \) is a subset of \( B^c \), \( P[A \cap B^c] = P[A] = \frac{3}{8} \). Furthermore, since \( \emptyset \) is a subset of \( B^c \), \( P[A \cup B^c] = P[B^c] = \frac{3}{4} \).

(b) The events \( A \) and \( B \) are dependent because
\[
P[AB] = 0 \neq \frac{3}{8} \cdot \frac{3}{2} = P[A] \cdot P[B]
\]

Problem 1.6.8

(a) Since \( C \) and \( D \) are independent \( P[CD] = P[C] \cdot P[D] \).
So,
\[
P[D] = \frac{P[CD]}{P[C]} = \frac{\frac{1}{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}
\]
In addition, \( P[C \cap D^c] = P[C] - P[C \cap D] = \frac{1}{2} - \frac{1}{2} = \frac{1}{2} \).
To find \( P[C^c \cap D^c] \), we first observe that
\[
P[C \cup D] = P[C] + P[D] - P[C \cap D] = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = \frac{1}{2}
\]
By De Morgan's law, \( C^c \cap D^c = (C \cup D)^c \). This implies
\[
P[C^c \cap D^c] = P[(C \cup D)^c] = 1 - P[C \cup D] = \frac{1}{2}
\]
Note that a second way to find \( P[C^c \cap D^c] \) is to use the fact that if \( C \) and \( D \) are independent, then \( C^c \) and \( D^c \) are independent. Thus
\[
P[C^c \cap D^c] = P[C^c] \cdot P[D^c] = (1 - P[C]) (1 - P[D]) = \frac{\sqrt{2}}{2}
\]
Finally, since \( C \) and \( D \) are independent events, \( P[C^c \cup D^c] = P[C^c] = \frac{\sqrt{2}}{2} \).
(b) Note that we found \( P[C \cup D] = 5/6 \). We can also use the earlier results to show:
\[
P[C \cup D] = P[C] + P[D] - P[C \cap D]
\]
\[
= \frac{1}{2} + \left(1 - \frac{2}{3}\right) - \frac{1}{6} = \frac{2}{3}.
\]

(c) By definition 1.7, events \( C \) and \( D^c \) are independent because
\[
P[C \cap D^c] = \frac{1}{6} = \left(\frac{1}{2}\right) \left(\frac{1}{3}\right) = P[C] \cdot P[D^c].
\]

Problem 1.6.9

For a sample space \( S = \{1, 2, 3, 4\} \) with equiprobable outcomes, consider the events
\[
A_1 = \{1, 2\}, \quad A_2 = \{2, 4\}, \quad A_3 = \{3, 4\}.
\]
Each event \( A_i \) has probability \( 1/2 \). Moreover, each pair of events is independent since
\[
P[A_1 A_2] = P[A_1] \cdot P[A_2] = \frac{1}{4}.
\]
However, the three events \( A_1, A_2, A_3 \) are not independent since
\[
P[A_1 A_2 A_3] = 0 \neq P[A_1] \cdot P[A_2] \cdot P[A_3].
\]