REAL ANALYSIS

QUALIFYING EXAM, SPRING 2014; CLOSED BOOKS

Solve four of the following five problems

1. Let $f : [0, 1] \to \mathbb{R}$ be a differentiable function.
   (i) Is it true that $f'$ is necessarily continuous? Justify your answer.
   (ii) Is it true that $f'$ is necessarily measurable? Justify your answer.

2. Compute $\int_{[0, \pi/2]} f \, d\mu$, where
   
   $f(x) = \begin{cases}
   \cos(x), & \text{if } \sin(x) \text{ is rational;} \\
   \cos^3(x), & \text{if } \sin(x) \text{ is irrational}
   \end{cases}
   $

3. Let $f : [a, b] \to \mathbb{R}$ be a measurable function such that $\int_a^b |f| \, d\mu = 0$. Show that $f(x) = 0$ a.e.

4. Let $f : [a, b] \to \mathbb{R}$ be a summable function and $\{c_n\}$ an increasing sequence such that $a < c_n < b$ for all $n \in \mathbb{N}$. Put $c := \lim_{n \to \infty} c_n$. Show that
   
   $\int_{c_1}^c f \, d\mu = \sum_{n=1}^{\infty} \int_{c_n}^{c_{n+1}} f \, d\mu.$

5. (i) Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function (i.e. $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$). Assume, in addition, that $f$ is continuous. Show that $f$ is linear.
   (ii) Is Statement (i) true if we replace continuity by measurability? Justify your answer.
REAL FUNCTIONAL ANALYSIS (MATH 6302)

QUALIFYING EXAM, SPRING 2014; CLOSED BOOKS

Solve four of the following five problems

1. Let $E$ be a Banach space. Show that $E$ is finite-dimensional if and only if every closed bounded subset of $E$ is compact.

2. Let $E$ be a normed space, $A, B \subset E$ and $A + B := \{a + b : a \in A, b \in B\}$.
   (i) Assume $A$ is open and $B$ is arbitrary. Show that $A + B$ is open.
   (ii) Assume $A$ is closed and $B$ is compact. Show that $A + B$ is closed.
   (iii) Assume $A$ is closed and $B$ is closed. Is $A + B$ necessarily closed? Justify your answer.

3. Let $H$ be the space of real continuous functions on $[-1, 1]$ equipped with the inner product $\langle f, g \rangle := \int_{-1}^{1} f(t)g(t)dt$. Let $K \subset H$ be the linear subspace of $H$ consisting of functions which are equal to zero at zero. Find the orthogonal complement of $K$ in $H$ (i.e., the linear space of functions from $H$ orthogonal to any function from $K$). Justify your answer.

4. Let $A : E \to F$ be a linear operator between two normed spaces. Show that $A$ is bounded if and only if there exists an open non-empty set $U \subset E$ such that $A(U)$ is bounded in $F$.

5. Let $c_0$ be the space of real sequences converging to zero and equipped with the sup-norm. Let $c_0^*$ stand for the conjugate (dual) space of linear continuous functionals on $c_0$ equipped with the standard norm: if $f \in c_0^*$, then:

\[ \|f\| = \sup_{\|x\| \leq 1} |f(x)|. \]

Show that $c_0^*$ is isometrically isomorphic to the space $l^1$ of real sequences $\{x_n\}$ with $\sum_n |x_n| < \infty$ equipped with the norm $\sum_n |x_n|$.
1. Let $A(t) \in C(-\infty, \infty)$ be periodic with period $T$, i.e. $A(t + T) = A(t)$.
   [ a ] (25 points) Prove that the fundamental set of solutions $\Phi(t)$ for the ODE $x' = A(t)x$ for $-\infty < t < \infty$ can be represented as $\Phi(t) = P(t)e^{tR}$, where $P(t)$ is a periodic nonsingular matrix with period $T$ and $R$ is a constant matrix.
   [ b ] (25 points) Find the fundamental set of solutions for $x'' = \sin(t)x'$. Find $R$ and explicitly show that the associated $2 \times 2$ matrix $P(t)$ for this problem is periodic.

2. (25 points) Suppose for a given continuous function $f(t)$ we find the equation

   $x' = \begin{pmatrix} -5 & 2 \\ -4 & 1 \end{pmatrix} x + f(t)$

   has a solution $\phi(t)$ which satisfies: $\sup\{|\phi(t)| : \tau \leq t < \infty\} < \infty$. Prove that all other solutions to above ODE must satisfy the same boundedness condition.

3. (25 points) Consider the boundary value problem on $[0, 2]$ for the ODE

   $Lx = \lambda x$, where $Lx = ix'$, and $x(2) = \alpha x(0)$

   with $\alpha$ being a constant complex number and $i = \sqrt{-1}$. Find the associated eigenvalues and eigenfunctions of this problem. For what values of $\alpha$ the eigenvectors are orthogonal. Please clearly justify your answer.
1. [a] (25 points) Determine all the equilibrium point of

\[ x'_1 = -x_2 + 2\sin x_1 \]
\[ x'_2 = x_1 - x_1 x_2. \]

[b] (25 points) Determine the stability of \( x = 0 \) for the nonlinear system given in part (a) by finding an appropriate Lyapunov function.

2. [a] (25 points) Analyze the stability properties of the equilibrium point \( x = 0 \) for the following equation.

\[ \mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 7 & 3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} (e^{x_1} - 1) \sin(x_2 t) \\ e^{-t} x_1 x_2 \end{pmatrix} \]

[b] (25 points) For the above nonlinear system, find the tangent spaces to its stable and unstable manifolds at the origin (if they exist).
Name: ________________________________

Abstract Algebra Qualifying Exam
Spring 2014
Tobias Hagge

April 9, 2014

Do as many as you are able. Show all work.

1. Let $G$ be a group. Show that if $g^2 = e$ for all $g \in G$, then $G$ is abelian.

2. Let $G$ be a group. Show that if $G/Z(G)$ is cyclic, then $G$ is abelian.

3. Show that if $(n_1, \ldots, n_k)$ is a sequence of relatively prime positive integers, then $\mathbb{Z}/n_1\mathbb{Z} \times \ldots \times \mathbb{Z}/n_k\mathbb{Z}$ is a cyclic group.

4. Let $G$ be a group of order 12. Show that either $G$ has a normal subgroup of order 3, or $G \cong A_4$ (hint: consider action by conjugation on the set of Sylow 3-subgroups).

5. Let $T$ be a the planar region bound by an equilateral triangle with barycenter at the origin. The symmetric group $S_3$ acts by reflections and rotations on $T$, such that the action by permutation $\sigma$ on $T$ reduces to permutation by $\sigma$ of the vertices. Up to isomorphism, is every transitive $S_3$-set present as an orbit of this action?

6. Show that no group of order 14 is simple.

7. Show that in a finite commutative ring $R$ with $1 \neq 0$, every prime ideal is maximal.

8. Let $R$ be a commutative ring with $1 \neq 0$. Suppose that every nonzero proper ideal of $R$ is maximal. Show that if $R$ has two distinct maximal ideals, then there are fields $F_1$ and $F_2$ such that $R \cong F_1 \oplus F_2$.

9. Show that if $R$ is a ring and $M$ is an irreducible $R$-module, then the ring $\text{End}_M(R)$ is a division ring.

10. Show that every module over a field is free.
Name: ____________________________

Complex Analysis Qualifying Exam
Spring 2014
Tobias Hagge

April 11, 2014

1. Let $a, b \in \mathbb{C}$, $k > 0$. Show that the set of points $\{z \in \mathbb{C} | |z - a| = k|z - b|\}$ is a line or circle.

2. Show that any meromorphic function on the plane with a finite limit at infinity is a rational function.

3. Prove the fundamental theorem of algebra using the argument principle. In particular, show that if $f$ is a polynomial of degree $n > 0$ and $D$ is a sufficiently large disk centered at the origin, $f$ contains $n$ zeros with multiplicity.

4. Show that every conformal homeomorphism of the upper half plane can be expressed in the form

$$f(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{R}$.

5. Compute the residue of the function $\cot z$ at $z = 0$.

6. Compute $\int_{0}^{\infty} \frac{\cos \pi}{x^2 + 1} \, dx$ using complex methods.
Name:

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

(1a) If

\[ A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \]

what are the singular values of \( A \)?

(b) If the matrix of left singular vectors is

\[ V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \]

what is the minimum length least squares solution \( x^+ \) to \( Ax = b \) if

\[ b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]?

(c) Prove the theoretical formula you used in Part (b) gives the minimum norm least squares solution.

(2) Assume that \( A \in \mathbb{R}^{n \times n} \) has \( n \) linearly independent eigenvectors \( v_1, \ldots, v_n \). Let \( \lambda_1, \ldots, \lambda_n \) denote the eigenvalues associated with the eigenvectors \( v_1, \ldots, v_n \). Show that the sequence \( (q_j) = (A^j q / \lambda_1^j) \) generated by the power method converges to the dominant eigenvector \( v_1 \) with convergence ratio \( r = |\lambda_2/\lambda_1| \), provided that \( |\lambda_1| > |\lambda_2| > |\lambda_3| \), and \( c_1 \neq 0 \) and \( c_2 \neq 0 \).
(3) Let

\[ A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}. \]

Find the QR factorization of \( A \) using

(a) Householder transforms,

(b) Givens rotations.

(c) Which of these two algorithms would be more efficient to use for this factorization if your matrix were large?

(d) Assuming \( b = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \), use one of your QR factorizations above to find the solution \( x \) to the system \( Ax = b \).

(4a) Show that for any induced matrix norm, \( \kappa(A) \geq 1 \).

(b) If \( Ax = b \) and \( (A + \delta A)(x + \delta x) = b \), prove the inequality

\[
\frac{||\delta x||}{||x + \delta x||} \leq \kappa(A) \frac{||\delta A||}{||A||}
\]

where \( \kappa(A) \) is the condition number of the matrix \( A \).

(c) Verify the inequality for the system

\[
\begin{bmatrix}
1 & 1/2 & 1/3 \\
1/2 & 1/3 & 1/4 \\
1/3 & 1/4 & 1/5 \\
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix} = \begin{bmatrix}
1 \\
1 \\
1 \\
\end{bmatrix}
\]

using

\[
\Delta A = \begin{bmatrix}
0 & 0 & 0.00003 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}.
\]

(d) Is the determinant of a matrix a good measure of the condition of a matrix? Give an example to help justify your answer.