Differential Equations

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Chapter 1

First Order Equations.

1.1 Separable Equations

A Differential Equation is Separable if it can be written as:

\[ f(x)dx = g(y)dy \]

The Solution is found by integrating both sides.

An Example: Solve:

\[ e^x ydx = (e^{2x} + 1)dy \quad y(0) = 1 \]

Solution:

\[ \int \frac{e^x}{e^{2x} + 1} dx = \int \frac{dy}{y} \]

Using the substitution \( u = e^x \), \( du = e^x dx \) on the integral on the left

\[ \int \frac{du}{u^2 + 1} = \int \frac{dy}{y} \]

\[ \arctan(u) + C = \ln |y| \quad \arctan(e^x) + C = \ln |y| \]

Applying our initial conditions

\[ \arctan(1) + C = \ln(1) \quad \frac{\pi}{4} + C = 0 \quad C = -\frac{\pi}{4} \]

\[ \arctan(e^x) - \frac{\pi}{4} = \ln |y| \]

Solving for \( y \) gives the solution to the differential equation:
An Application:

A parachutist falling toward Earth is subject to two forces: the parachutist weight \( w = 32m \) and the drag of the parachute. The drag of the parachute the drag is proportional to the velocity of the parachute and in this case is equal to \( 8|v| \). The parachutist weight is 128lb and initial velocity is zero. Find formulas for the parachutists velocity \( v(t) \) and distance \( x(t) \).

Since \( x(t) \) increases as the parachutist falls, the downward direction is the positive direction. The force from the parachutist’s weight acts in the positive direction while the drag from the parachute acts in the negative direction. Since the parachutist falls down (the positive direction) velocity is always positive so \( |v| = v \). The resultant force will be the force of the weight of the parachutist minus the force of the drag of the parachute:

\[
F = 128 - 8v
\]

And the mass of the parachutist is:

\[
128 = 32m \quad m = 4
\]

By Newton’s second law:

\[
F = ma = 4 \frac{dv}{dt}
\]

since \( a = \frac{dv}{dt} \)

This gives a differential equation:

\[
4 \frac{dv}{dt} = 128 - 8v
\]

This equation is Separable:

\[
\frac{dv}{32 - 2v} = dt
\]

Integrating

\[
-\frac{1}{2} \ln(32 - 2v) = t + C \quad \text{or} \quad \ln(32 - 2v) = -2t + C
\]

Applying the initial condition \( v(0) = 0 \)

\[
C = \ln(32)
\]

Solving for \( v(t) \) gives:

\[
v(t) = 16 - 16e^{-2t}
\]

The parachutist terminal velocity is given by:
1.1. SEPARABLE EQUATIONS

\[ \lim_{t \to \infty} v(t) = \lim_{t \to \infty} 16 - 16e^{-2t} = 16\text{ft/sec} \]

We can now find an equation: \( x(t) \) for how far the parachutist has fallen:

\[ x(t) = \int v(t) dt = \int (16 - 16e^{-2t}) dt = 16t - 8e^{-2t} + K \]

Since \( x(0) = 0 \) we see that \( K = -8 \). Thus,

\[ x(t) = 16t - 8e^{-2t} - 8 \]

1. Show

\[ y = \cosh(x) = \frac{e^x + e^{-x}}{2} \]

is a solution to

\[ K = \frac{1}{y^2} \]

where \( K \) is the curvature

\[ K = \frac{|y''|}{\left(1 + (y')^2\right)^{\frac{3}{2}}} \]

2. Find the values of \( r \) so that \( y = e^{rx} \) is a solution to

\[ y''' - 6y'' + 11y' - 6y = 0 \]

3. Find the values of \( k \) so that \( y = \sin(kx) \) is a solution to

\[ y'' + 100y = 0 \]

4. Find the values of \( n \) so that \( y = x^n \) is a solution to

\[ x^2y'' + 7xy' + 8y = 0 \]

5. The Clairaut Equation is

\[ y = xy' + f(y') \]

Show \( y = kx + f(k) \) is a solution for some constant \( k \).

Use the above result to find a solution to
\[ y = xy' + (y')^3 \]

6. Show

\[ y = \begin{cases} 
  e^x - 1 & x \geq 0 \\
  1 - e^{-x} & x < 0 
\end{cases} \]

is a solution to

\[ y' = |y| + 1 \]

Remember, you must use the definition of the derivative to calculate \( y'(0) \).

7. Solve the differential equation

\[ (x^2 + 1)dy = (4x + xy^2)dx \quad y(0) = 2 \]

8. Solve the differential equation

\[ (2xy + 2x)dx = e^{-x^2}dy \quad y(\sqrt{\ln(5)}) = 0 \]

9. Solve the differential equation

\[ \frac{dy}{dx} = \sqrt{16x^2 - 4x^2y^2} \quad y(2) = 1 \]

10. Solve the differential equation

\[ \frac{dy}{dx} = \ln \left( x + \sqrt{x^2 - 1} \right)^y \quad y(1) = e \]

11. Solve the differential equation

\[ xdy = (y^2 + 4y + 5)\sqrt{x^3 - 1}dx \quad y(0) = -2 \]

12. Solve the differential equation

\[ (x^3 - x^2 + x - 1)dy = (3x^2y - 2xy + y)dx \quad y(0) = e \]

13. Solve the differential equation

\[ \sqrt{y + xy}dy = \arcsin(x)dx \quad y(0) = 1 \]
14. Solve the differential equation
\[
\cos(2x)dy = (1 + \sin(2x))(\cos(x) - \sin(x))(1 + y^2)dx
\]
\[
y\left(\frac{\pi}{4}\right) = 0
\]

15. Solve the initial value problem
\[
\sqrt{1-x^2} = 2xy\frac{dy}{dx}, \quad y(1) = 2
\]

16. Solve the initial value problem
\[
4x \ln(x) + 4xy \ln(x) = y\frac{dy}{dx}, \quad y(e) = 0
\]

17. Solve the initial value problem
\[
15e^{-y}\sin^3(x) = \cos^6(x)\frac{dy}{dx}, \quad y(0) = \ln(2)
\]

18. Solve the initial value problem
\[
1 + \sqrt{y} = (1 + \sin(x))\frac{dy}{dx}, \quad y(0) = 0
\]

19. Solve
\[
(e^x - e^{-x})(y^2 + 1) = (2ye^x + 2ye^{-x})\frac{dy}{dx}
\]

20. Solve
\[
(y^2 + 1)dx = (x^{\frac{1}{4}} + x^{\frac{3}{4}})dy
\]

21. Solve:
\[
\frac{dx}{\sin(x)} = \frac{dy}{\cos^2(x) + \cos^2(x)\sqrt{y}}
\]

22. Solve:
\[
x^2(y + \sqrt{y})dx = (x^4 + 2x^2 + 1)dy
\]

23. Solve:
CHAPTER 1. FIRST ORDER EQUATIONS.

\[ \frac{dy}{dx} = \frac{xe^x(y^2 + 4y + 5)}{x^2 + 2x + 1} \quad y(0) = -1 \]

24.
Solve

\[ \frac{xe^{3x}}{y} = 2(3x + 1)^2 \frac{dy}{dx} \]

25.
Solve the differential equation

\[ \frac{\sqrt{x} - 2}{\sqrt{y} - 1} \frac{dy}{dx} = \frac{\sqrt{y} + 1}{\sqrt{x} + 2} \frac{dx}{dy} \]

26.
Solve the differential equation

\[ (xy + 2x + y + 2)dx = (x^2y^2 + 2xy^2)dy \]

27.
Solve the differential equation

\[ \frac{4}{xy + xy^2 + x^2y + x^2y^2} \frac{dy}{dx} = 1 \]

28.
Use the Second Fundamental Theorem of Calculus to verify

\[ y = Ce^{-\int g(u)du} \]

is a solution to

\[ y' + g(x)y = 0 \]

29.
Use the substitution \( u = ye^x \) to transform the equation into a separable equation and then solve it

\[ ydx + (1 + ye^{2x})dy = 0 \]

30.
Use the substitution \( y = ze^x \) to transform the equation into a separable equation and then solve it

\[ \frac{dy}{dx} = y + \sqrt{e^{2x} - y^2} \]

31.
Use the substitution \( z = y^2 + x - 1 \) to transform the equation into a separable equation and then solve it

\[ 2y \frac{dy}{dx} = y^2 + x - 1 \]
Some times it is useful to convert a differential equation to polar coordinates before solving it. The conversions to polar coordinates is:

\[ x = r \cos \theta \quad y = r \sin \theta \]

Calculating the total differential of both \( x \) and \( y \) we get:

\[ dx = \cos \theta dr - r \sin \theta d\theta \quad dy = \sin \theta dr + r \cos \theta d\theta \]

Making

\[ \frac{dy}{dx} = \frac{\sin \theta dr + r \cos \theta d\theta}{\cos \theta dr - r \sin \theta d\theta} \]

Use this conversion to polar coordinates to solve the next two problems:

32. Solve by converting to polar coordinates

\[ x + y = x \frac{dy}{dx} \]

33. Solve by converting to polar coordinates

\[ (2xy + 3y^2)dx = (2xy + x^2)dy \]

34. Solve by converting to polar coordinates

\[ \frac{dy}{dx} = \frac{y^3 + x^2y - x - y}{x^3 + xy^2 - x + y} \]

35. Salt water containing .25 pounds of salt per gallon is being pumped into a tank initially containing 100 gallons of water and 10 pounds of salt at a rate of 4 gallons per minute. The mixture in the tank is kept well stirred and fluid flows out of the tank at a rate of 4 gallons per minute. Find a formula that represents the amount of salt in the tank at any time.

36. The logistic differential equation that models the size of a population of species in an environment of fixed size is given by the following differential equation:

\[ \frac{dP}{dt} = kP(M - P) \]

where \( M \) is the carrying capacity of the environment: that is, \( M \) is the maximum population of the species that can fit in the environment and \( k > 0 \) is a constant depending on the reproduction rate of the species. For example if \( P \) represent the population of a bacteria in a petri dish then \( M \) would be the maximum population of the bacteria in the dish. We also see from the differential equation if a population \( P \) is less than \( M \) then \( \frac{dP}{dt} > 0 \) and the population will increase and approach the carrying capacity and if \( P \) is greater than \( M \) then \( \frac{dP}{dt} < 0 \) and the population will decrease and approach the carrying capacity.
CHAPTER 1. FIRST ORDER EQUATIONS.

Show the population is increasing fastest when the population is half the carrying capacity then solve this differential equation for the population $P(t)$ as a function of time and show:

$$\lim_{t \to \infty} P(t) = M$$

37.

It has been calculated that the world population cannot exceed 20 billion people. In 1970 the population was 3.7 billion and in 2014 the population grew to 6.8 billion. Write a logistic differential equation representing the world population and solve it for the world population as a function of time. When will the population exceed 10 billion?

38.

Another type of population model is the Gopertz growth model. It is similar to the logistic equation in that the model assumes the population will increase at a rate proportional to the size of the population. That means the population will increase at a rate of $kP(t)$. The like the logistic model the Gopertz growth model also takes into account the maximum population a species can have in an environment of fixed size and resources. Instead of using $(M - P(t))$ as a factor like the logistic model does the Gopertz growth model uses $\ln \left( \frac{M}{P(t)} \right)$ as a factor, with $M$ being the maximum population (carrying capacity). The Gopertz growth model is

$$\frac{dP}{dt} = kP \ln \left( \frac{M}{P} \right) \quad k > 0$$

We also see from the differential equation if a population $P$ is less than $M$ then $\frac{dP}{dt} > 0$ and the population will increase and approach the carrying capacity and if $P$ is greater than $M$ then $\frac{dP}{dt} < 0$ and the population will decrease and approach the carrying capacity.

Show the population is increasing fastest when the population is $\frac{M}{e}$ and then solve this differential equation for the population $P(t)$ as a function of time and show:

$$\lim_{t \to \infty} P(t) = M$$

39.

200 fish of a particular species are introduced to a lake which can sustain no more than 3000 fish. After 2 years the fish population had increased to 800 fish. If the population follows the Gopertz growth model how long after the introduction of the fish to the lake will the population reach 2000 fish. Repeat the calculation using the logistic model and compare the results.

40.

Differential equations can also be used to model the genetic change or evolution of a species. A commonly used hybrid selection model is

$$\frac{dy}{dt} = ky(1 - y)(a - by)$$

With $y$ represents the portion of a population that has a certain characteristic and $a, b, k$ constants and $t$ is time measured in generations.

At the beginning of a study of a population of a particular species it is found that half population had the advantageous trait $T$ and three generations later 60 percent of the population had trait $T$. Use the hybrid selection model with $a = 2$ and $b = 1$ to determine the number of generations it will take until more than 80 percent of the population has trait $T$. 
41. Newton’s law of cooling states that an object with temperature $T$ in a medium of constant temperature $M$ will experience a change in temperature proportional to the difference in the temperature of the object and the medium $(M - T)$. This gives the differential equation:

$$\frac{dT}{dt} = k(M - T)$$

A cup of 170° coffee is placed in a 75° room. After 10 minutes the coffee is measured to have a temperature of 150°. How long will it take for the coffee to cool to 120°?

42. In the study of learning it has been shown that a person’s ability to learn a task is governed by the differential equation

$$\frac{dy}{dt} = \frac{2py^2}{\sqrt{n}} \left(1 - \frac{y}{n}\right)^{\frac{3}{2}}$$

Where $y$ represents the level that a student has learned a skill as a function of time and $n$ and $p$ are constants depending on the person learning the skill and the difficulty of learning the skill. Solve this differential equation with the initial condition $y(0) = 0$ and $p = 1$, $n = 4$.

43. Torricelli’s Law states that water draining from a tank of volume $V(t)$ through a hole of area $a$ in the bottom will have an exiting velocity of

$$v(t) = \sqrt{2gy(t)}$$

where $y(t)$ measures the height of the water level above the hole in the tank. The change in volume in the tank is given by

$$\frac{dV}{dt} = -av(t) = -a\sqrt{2gy(t)}$$

If $A(y)$ denotes the area of the cross section of the tank at height $y$ then for any slice of water at a height of $\gamma$ and thickness $d\gamma$ will have volume

$$V(y) = \int_0^y A(\gamma) d\gamma$$

Using the second fundamental theorem of calculus to differentiate this integral gives

$$\frac{dV}{dt} = A(y) \frac{dy}{dt}$$

equating this result to the previous formula for $\frac{dV}{dt}$ gives Torricelli’s Law:

$$A(y) \frac{dy}{dt} = -a\sqrt{2gy(t)}$$

Use the above results to find how long it takes a spherical tank with radius of 60 inches to be drained through a 1 inch hole in the bottom.

44. Show that if $y_1$ and $y_2$ are solutions to
CHAPTER 1. FIRST ORDER EQUATIONS.

\[ y' + P(x)y = Q_1(x) \]

and

\[ y' + P(x)y = Q_2(x) \]

respectively then \( y = y_1 + y_2 \) is a solution to

\[ y' + P(x)y = Q_1(x) + Q_2(x) \]

45. Show that if \( y_1 \) and \( y_2 \) are solutions to

\[ y' + P(x)y^2 = Q_1(x) \]

and

\[ y' + P(x)y^2 = Q_2(x) \]

respectively then \( y = y_1 + y_2 \) is not a solution to

\[ y' + P(x)y^2 = Q_1(x) + Q_2(x) \]

46. There are about 3300 families of human languages spoken in the world. Assuming that all languages have evolved from a single language and that one family of languages evolves into 1.5 families of language every 6000 years how long ago was the original language spoken?

47. For every point \( P(x, y) \) on a curve in the first quadrant, the rectangle containing the points \( O(0, 0) \) and \( P(x, y) \) as vertices is divided by the curve into two regions: upper region A and lower region B. If the curve contains the point \( Q(1, 3) \), and region A always has twice the area of region B, find the equation of the curve.

48. Find a function \( f(x) \) with the following properties: The average value of \( f \) on \([1, x] \) is equal to twice the functions value at \( x \) and \( f(2) = 1 \).

1.2 First Order Linear Equations

A Differential Equation is First Order Linear if it has the form:

\[ \frac{dy}{dx} + P(x)y = Q(x) \]

To solve this equation we recognize the left hand side: \( \frac{dy}{dx} + P(x)y \) looks close to the derivative of the product of some function times \( y \). Idea: multiply both sides of the equation by some function \( I(x) \) to make the left hand side the derivative of the product of \( I(x) \) times \( y \). Multiplying both sides by \( I(x) \) gives:

\[ I(x) \frac{dy}{dx} + I(x)P(x)y = I(x)Q(x) \]
If the left hand side is the derivative of the product \( I(x) \cdot y \):

\[
\frac{d}{dx} \left( I(x) \cdot y \right) = I(x) \cdot \frac{dy}{dx} + I'(x) \cdot y
\]

Then:

\[
I(x) \frac{dy}{dx} + I(x)P(x) \cdot y = I(x) \cdot \frac{dy}{dx} + I'(x) \cdot y
\]

So

\[
I(x)P(x) \cdot y = I'(x) \cdot y
\]

\[
\frac{I'(x)}{I(x)} = P(x)
\]

Integrating gives:

\[
\ln |I(x)| = \int P(x)dx
\]

Solving for the Integrating Factor \( I(x) \) gives:

\[
I(x) = e^{\int P(x)dx}
\]

After multiplying both sides of the original differential equation by \( I(x) \) the left hand side is the derivative of the product \( I(x) \cdot y \) so the equation:

\[
I(x)\frac{dy}{dx} + I(x)P(x) y = I(x)Q(x)
\]

Becomes:

\[
\frac{d}{dx} \left( I(x) \cdot y \right) = I(x)Q(x)
\]

Integrating gives:

\[
\left( I(x) \cdot y \right) = \int I(x)Q(x)dx
\]

And the solution is given by:

\[
y = \frac{1}{I(x)} \left( \int I(x)Q(x)dx + C \right)
\]

An Example: Solve:

\[
\cos(x) \frac{dy}{dx} + \sin(x) y = \sec^2(x)
\]

Writing the differential equation in standard form:

\[
\frac{dy}{dx} + \tan(x)y = \sec^3(x)
\]

Creating the integrating factor
\[ I = e^{\int \tan(x) \, dx} = e^{\ln(\sec(x))} = \sec(x) \]

Our Solution is:

\[ y = \frac{1}{\sec(x)} \left( \int \sec(x) \sec^3(x) \, dx + C \right) \]
\[ y = \cos(x) \left( \int \sec^4(x) \, dx + C \right) = \cos(x) \left( \int \sec^2(x)(1 + \tan^2(x)) \, dx + C \right) \]
\[ y = \cos(x) \left( \int \sec^2(x) \, dx + \int \tan^2(x) \sec(x) \, dx + C \right) \]

Using the substitution \( u = \tan(x) \), \( du = \sec^2(x) \, dx \) on the second integral

\[ y = \cos(x) \left( \int \sec^2(x) \, dx + \int u^2 \, du + C \right) \]
\[ y = \cos(x) \left( \tan(x) + \frac{u^3}{3} + C \right) \]
\[ y = \cos(x) \left( \tan(x) + \frac{1}{3} \tan^3(x) + C \right) \]

\[ \blacksquare \]

**Example**

In this next example we will transform a nonlinear differential equation into a linear equation by converting it to polar coordinates.

\[(x^2 + y^2 + x) \frac{dy}{dx} = y\]

Let \( x = r \cos \theta \), \( y = r \sin \theta \)

Therefore \( dx = \cos \theta \, dr - r \sin \theta \, d\theta \) and \( dy = \sin \theta \, dr + r \cos \theta \, d\theta \)

Under this substitution our differential equation becomes:

\[(r^2 + r \cos \theta)(\sin \theta \, dr + r \cos \theta \, d\theta) = r \sin \theta (\cos \theta \, dr - r \sin \theta \, d\theta)\]

Multiplying things out gives

\[ r^2 \sin \theta \, dr + r^3 \cos \theta \, d\theta + r \cos \theta \sin \theta \, dr + r^2 \cos^2 \theta \, d\theta = r \sin \theta \cos \theta \, dr - r^2 \sin^2 \theta \, d\theta \]

Which reduces to

\[ r \cos \theta \, d\theta + \sin \theta \, dr + d\theta = 0 \]
\[
\frac{dr}{d\theta} + \cot \theta r = -\csc \theta
\]

This is first order linear with integrating factor

\[
I = e^{\int \cot \theta d\theta} = \sin \theta
\]

And the solution is...

\[
r = \frac{1}{\sin \theta} \left( - \int \sin \theta \csc \theta d\theta + C \right)
\]

which reduces to

\[
r = \frac{C - \theta}{\sin \theta}
\]

Or

\[
r \sin \theta = C - \theta
\]

Converting back to rectangular coordinate system gives the solution:

\[
y = C - \arctan \left( \frac{y}{x} \right)
\]

49. Solve:

\[
xy' - 3y = x^4 \quad y(1) = 1
\]

50. Solve:

\[
y' + e^x y = e^x \quad y(0) = 2e
\]

51. Solve:

\[
y' + \tan(x)y = \tan(x) \quad y \left( \frac{\pi}{4} \right) = 1
\]

52. Solve:

\[
\sqrt{1 - x^2} \frac{dy}{dx} + y = 1 \quad y(0) = 4
\]

53. Solve:

\[
\frac{dy}{dx} + \frac{6x^2 - 4x + 8}{x^3 - x^2 + 4x - 4} y = \frac{e^{x^3 + 12x}}{(x - 1)^2(x^2 + 4)}
\]
54. Solve:
\[ \frac{dy}{dx} + \frac{\cos(x) - \sin(x)}{\cos(x) + \sin(x)}y = \sec^3(x) \quad y(0) = 4 \]

55. Solve:
\[ (1 - x^2) \frac{dy}{dx} - xy = 1 \quad y\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2} \]

56. Solve:
\[ \frac{dy}{dx} + \frac{4x + 1}{x}y = e^x \quad y(1) = 0 \]

57. Solve:
\[ (1 + x^2) \frac{dy}{dx} + (4x^2 - 4x + 2)y = 9 \ln(x) \quad y(1) = 0 \]

58. Solve:
\[ \frac{dy}{dx} + \sin(x)y = \sin(2x) \]

59. Solve:
\[ \frac{dy}{dx} + \frac{y}{1 + e^{-x}} = \frac{1}{e^{2x} + 2xe^x + x^2} \quad y(0) = 0 \]

60. Solve:
\[ \frac{dy}{dx} - 2xy = (2 + x^{-2}) \quad y(1) = 0 \]

61. Solve:
\[ (1 + x^2) \frac{dy}{dx} - 2xy = (1) \quad y(0) = 1 \]

62. Solve:
\[ \cos^2(x)y' + y = 1 \quad y(0) = -3 \]

63. Solve:
1.2. FIRST ORDER LINEAR EQUATIONS

\[ \sin(x) \frac{dy}{dx} + \cos(x)y = \ln(x) \]

64. Solve:

\[ (e^x + e^{-x}) \frac{dy}{dx} + \left( \frac{(e^x + e^{-x})^2}{(e^x - e^{-x})} \right) y = 1 \]

65. Solve:

\[ (1 + x^4) \frac{dy}{dx} - 4x^3 y = (x^5 + x) \arctan(x^2) \quad y(1) = \pi \]

66. Solve:

\[ (1 + x^2) \frac{dy}{dx} + xy = (x^2 + 1)^2 \quad y(0) = 1 \]

67. Solve:

\[ (1 + x^2) \frac{dy}{dx} - 4xy = x^2 \]

68. Solve:

\[ \frac{dy}{dx} + \frac{6x}{x^4 + 5x^2 + 4} y = x \]

69. Solve:

\[ y - \frac{x}{x} \frac{dy}{dx} = y^2 e^y \frac{dy}{dx} \]

70. Solve:

\[ \frac{dy}{dx} + \frac{2 + \tan^2(x)}{x + \tan(x)} y = \cos(x) \]

71. Solve:

\[ \frac{dy}{dx} + (3x^2 + 2x) y = 3x^5 + 5x^4 + 2x^3 \]

72. Solve:

\[ \frac{dy}{dx} + \frac{1 + \cos^3(x)}{\sin(x) \cos(x)(1 + \cos(x))} y = \cos^2(x) \]

73. Find all values of \( k \) so that the solution \( y \) approaches 0 as \( x \) approaches \( \infty \)
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\[ y' + \frac{k}{x}y = x^2 \]

74. Express the solution to

\[ \frac{dy}{dx} = 1 + 2xy \]

in terms of the error function:

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \]

75. The solution to the differential equation

\[ \frac{dy}{dx} + P(x)y = Q(x) \]

is

\[ y(x) = Ce^{-2x} + t + 1 \]

Find functions \( P(x) \) and \( Q(x) \).

76. Salt water containing .25 pounds of salt per gallon is being pumped into a tank initially containing 100 gallons of water and 10 pounds of salt at a rate of 4 gallons per minute. The mixture in the tank is kept well stirred and fluid flows out of the tank at a rate of 2 gallons per minute. Find a formula that represents the amount of salt in the tank at any time.

77. At \( t = 0 \) one unit of a drug is administered to a patient who is hooked up to an IV drip supplying him with more of the drug so that one unit of the drug is always present in his system. If the patient’s liver removes 15 percent of the drug each hour how much of the drug must be administered by the IV each hour to keep 1 unit of the drug present?

78. The following equation is not separable or linear. Use the substitution \( u = e^{2y} \) to transform it into a linear equation and solve it.

\[ 2xe^{2y} \frac{dy}{dx} = 3x^4 + e^{2y} \]

79. The following equation is not separable or linear. Use the substitution \( u = e^y \) to transform it into a linear equation and solve it.

\[ \frac{dy}{dx} + \frac{2}{x} = \frac{e^{-y}}{1 + x^3} \]

80. The following equation is not separable or linear. Use the substitution \( u = \tan(y) \) to transform it into a linear equation and solve it.
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\[ \sec^2(y) \frac{dy}{dx} - \frac{3}{x} \tan(y) = x^4 \]

81. The following equation is not separable or linear. Use the substitution \( u = \frac{1}{4+y} \) to transform it into a linear equation and solve it.

\[ \frac{1}{(4+y)^2} \frac{dy}{dx} - \frac{2}{x(4+y)} = \ln(x) \]

82. The following equation is not separable or linear. Use the substitution \( y = e^u \) to transform it into a linear equation and solve it.

\[ x \frac{dy}{dx} - 4x^2y + 2y \ln(y) = 0 \]

83. The following equation is not separable or linear. Use the substitution \( z = \ln(y) \) to transform it into a linear equation and solve it.

\[ x \frac{dy}{dx} + 2y \ln(y) = 4x^2y \]

84. The following equation is not separable or linear. Use the substitution \( z = \ln(y) \) to transform it into a linear equation.

\[ \frac{dy}{dx} + f(x)y = g(x)y \ln(y) \]

85. The following equation is not separable or linear. Use the substitution \( u = y^2 \) to transform it into a linear equation and solve it.

\[ 2xy \frac{dy}{dx} + 2y^2 = 3x - 6 \]

86. The differential equation governing the velocity \( v \) of a falling object subject to air resistance is

\[ m \frac{dv}{dt} = mg - kv \quad k > 0 \quad v(0) = v_0 \]

Solve this differential equation and determine the limiting velocity of the object.

87. If \( A(t) \) represents the amount of money in an account then the change in the amount in the account is given by:

\[ \frac{dA}{dt} = \text{Deposits} - \text{Withdraws} + \text{Interest} \]

With a constant interest rate \( r \) the interest on the account is \( rA \) (remember \( A \) is the amount of money you will be getting interest on). This makes the differential equation:

\[ \frac{dA}{dt} = \text{Deposits} - \text{Withdraws} + rA \]
In first order linear form:
\[
\frac{dA}{dt} - rA = \text{Deposits} - \text{Withdraws}
\]

Use this differential equation to solve the following:

A person opens an account yielding 3 percent interest is opened with an initial investment of 1000 dollars. On the first year they deposit 100 dollars, the year month 110 dollars, the third year 120 dollars. So each year they deposit 10 dollars more than the year before. How much will they have in the account in ten years?

An equation of the form:
\[
y' + P(x)y = 0
\]
is called a first order linear homogenous (Q(x)=0) differential equation. It can be solve by separation of variables while
\[
y' + P(x)y = Q(x)
\]
cannot. Show that if \(y_h(x)\) is the solution to the homogenous equation
\[
y' + P(x)y = 0
\]
then
\[
y(x) = y_h(x) \int \frac{Q(x)}{y_h(x)} \, dx
\]
is the solution to the nonhomogenous equation
\[
y' + P(x)y = Q(x)
\]
Use this technique to solve:
\[
xy' + y = e^{4x}
\]
The Riccati Differential Equation is an equation of the form:
\[
\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x)
\]
If \(u(x)\) is a solution to the equation then the substitution \(y = u + \frac{1}{v}\) will transform the equation into a first order linear equation.

Solve the Riccati Equation:
\[
\frac{dy}{dx} = -8xy^2 + 4x(4x + 1)y - (8x^3 + 4x^2 - 1)
\]
\(u(x) = x\) is one solution
Solve the Riccati Equation:

\[
\frac{dy}{dx} = x^3(y - x)^2 + \frac{y}{x} \quad u(x) = x \quad \text{is one solution}
\]

91.

Show the nonlinear differential equation

\[(y')^2 + y \cdot y' = x^2 + xy\]

can be factored into

\[(y' + y + x)(y' - x) = 0\]

Set each factor to zero and solve each of the differential equations. Then show each solution you obtain is also a solution to the original differential equation.

### 1.3 Bernoulli Equation

The Bernoulli Differential Equation is a nonlinear equation of the form:

\[
\frac{dy}{dx} + P(x)y = Q(x)y^n
\]

After the substitution \(z = y^{1-n}\) the Bernoulli Equation will be First Order Linear. Differentiating gives:

\[
\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}
\]

Multiplying both sides of the Bernoulli equation by \((1-n)y^{-n}\) gives:

\[(1-n)y^{-n}\frac{dy}{dx} + (1-n)P(x)y^{1-n} = (1-n)Q(x)\]

Under the substitution this equation becomes:

\[
\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)
\]

Which is a First Order Linear differential equation.

**An Example:** Solve:

\[
\frac{dy}{dx} + 2xy = e^{3x^2 + 2x}y^4
\]

Let \(z = y^{-4} = y^{-3}\) \quad So \(\frac{dz}{dx} = -3y^{-4}\frac{dy}{dx}\)

Multiplying both sides of the differential equation by \(-3y^{-4}\) gives:

\[-3y^{-4}\frac{dy}{dx} + (-6x)y^{-3} = -3e^{3x^2 + 2x}\]

Under our substitution our differential equation becomes:

\[
\frac{dz}{dx} + (-6x)z = -3e^{3x^2 + 2x}
\]
This is First Order Linear. Our Integrating Factor is:

\[ I = e^{\int (-6x) \, dx} = e^{-3x^2} \]

The solution is:

\[ z = \frac{1}{e^{-3x^2}} \left( \int e^{-3x^2} (-3e^{3x^2+2x}) \, dx + C \right) \]

\[ z = e^{3x^2} \left( -3 \int e^{2x} \, dx + C \right) = e^{3x^2} \left( \frac{-3}{2} e^{2x} + C \right) \]

Back substituting gives:

\[ \frac{1}{y^3} = \frac{-3}{2} e^{3x^2+2x} + Ce^{3x^2} = \frac{Ce^{3x^2} - 3e^{3x^2+2x}}{2} \]

\[ y^3 = \frac{2}{Ce^{3x^2} - 3e^{3x^2+2x}} \]

So

\[ y = \sqrt[3]{\frac{2}{Ce^{3x^2} - 3e^{3x^2+2x}}} \]

92.
Solve:

\[ y \frac{dy}{dx} + \frac{xy^2}{x^2+1} = x \]

93.
Solve:

\[ \frac{dy}{dx} + \cot(x)y = \cos^3(x)\sqrt{y} \]

94.
Solve:

\[ x \frac{dy}{dx} - \frac{y}{2} = x \arcsin(x) y^5 \]

95.
Solve:

\[ \frac{dy}{dx} = y^4 \cos(x) + y \tan(x) \]

96.
Solve:

\[ \frac{dy}{dx} + \frac{9x + 2}{9x^2 + 4x} y = \ln(x)y^{-1} \]
97. Solve:
\[ \frac{dy}{dx} + \frac{x}{2 + 2x^2} y = \frac{1}{xy} \]

98. Solve:
\[ \frac{dy}{dx} - \frac{\sec(x) \tan(x)}{1 + \sec(x)} y = \frac{\sin^3(x)}{\cos^4(x)} y^2 \]

99. Solve:
\[ 4 \cos^2(x) \frac{dy}{dx} + y = 4 \sin(x) \cos(x)y^{-3} \]

100. Solve:
\[ \frac{dy}{dx} + \frac{1}{\sin(2x)} y = (1 - \cos(2x))y^3 \]

101. Solve:
\[ \frac{dy}{dx} + \frac{\cos(x)}{4 + 4 \sin(x)} y = \cos^3(x)y^{-3} \]

102. Solve:
\[ x \frac{dy}{dx} - y = e^{x^2} y^5 \]

103. Solve:
\[ \frac{dy}{dx} + \frac{5x}{x^2 + 1} y = (x^2 + 1)y^2 \]

104. Solve:
\[ x \frac{dy}{dx} + y = -y^2 \]

105. Solve:
\[ \frac{dy}{dx} + \tan(x)y = \sec^4(x)y^3 \]

106. Solve:
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\[ \frac{2}{dx} \frac{dy}{dx} + \frac{\cos(x)}{1 + \sin(x)} y = \cos^3(x)y^{-1} \]

107.

Solve:

\[ \frac{dy}{dx} + \frac{\cos(x) - \sin(x)}{3 \cos(x) + 3 \sin(x)} y = (\cos(x) - \sin(x))y^{-2} \]

108.

Solve:

\[ \frac{dy}{dx} = \frac{1}{xy + x^2y^3} \]

109.

Use the substitution \( u = e^y \) to transform the equation into a Bernoulli equation and then solve it

\[ \frac{dy}{dx} + \frac{1}{x} = 6xe^{3y} \]

Another differential equation that can be transformed into a first order linear differential equation is Lagrange’s Equation. This is an equation of the form:

\[ y = xF(y') + G(y') \]

To simplify notation let \( y' = p \) and our equation becomes

\[ y = xF(p) + G(p) \]

Differentiating with respect to \( x \) gives

\[ y' = xF'(p) \frac{dp}{dx} + F(p) + G'(p) \frac{dp}{dx} \]

Or

\[ p = xF'(p) \frac{dp}{dx} + F(p) + G'(p) \frac{dp}{dx} \]

Solving for \( \frac{dx}{dp} \)

\[ p - F(p) = \left( xF'(p) + G'(p) \right) \frac{dp}{dx} \]

\[ \frac{dp}{dx} = \frac{p - F(p)}{xF'(p) + G'(p)} \]

\[ \frac{dx}{dp} = \left( \frac{F'(p)}{p - F(p)} \right) x + \left( \frac{G'(p)}{p - F(p)} \right) \]

In standard form

\[ \frac{dx}{dp} + \left( \frac{F'(p)}{F(p) - p} \right) x = \left( \frac{G'(p)}{p - F(p)} \right) \]

This is now a first order linear equation.
110. Solve the Lagrange Equation

\[ y = 2xy' + 4(y')^3 \]

You may leave your solution in parametric form with \( p \) the parameter.

111. Solve the Lagrange Equation

\[ y = 2xy' - 9(y')^2 \]

You may leave your solution in parametric form with \( p \) the parameter.

### 1.4 Homogenous Equation

A differential equation is Homogenous if it has the form:

\[ \frac{dy}{dx} = f \left( \frac{y}{x} \right) \]

The substitution \( z = \frac{y}{x} \) or \( y = xz \) will transform the Homogenous equation into a Separable equation.

\[ \frac{dy}{dx} = z + x \frac{dz}{dx} \]

Under this substitution the Homogenous equation becomes:

\[ z + x \frac{dz}{dx} = f(z) \]

This reduces to the Separable equation

\[ \frac{dz}{f(z) - z} = \frac{dx}{x} \]

**An Example:** Solve

\[ \frac{dy}{dx} = \frac{y^3 + 2x^2y}{xy^2 + x^3} \]

**Solution:**

Multiplying the numerator and denominator on the right hand side by \( \frac{1}{x^3} \) gives:

\[ \frac{dy}{dx} = \frac{y^3 + 2y}{y^2 + 1} \]

Let \( z = \frac{y}{x} \) so \( y = xz \) \( \frac{dy}{dx} = z + x \frac{dz}{dx} \)

Under this substitution our differential equation becomes:

\[ z + x \frac{dz}{dx} = \frac{z^3 + 2z}{z^2 + 1} \]
\[ \frac{dz}{dx} = \frac{z}{z^2 + 1} \]
\[ \frac{z + 1}{z} \, dz = \frac{dx}{x} \]
\[ \int \left( \frac{z + 1}{z} \right) \, dx = \int \frac{dx}{x} \]
\[ \frac{z^2}{2} + \ln(z) = \ln(x) + C \]
\[ \frac{1}{2} \frac{y^2}{x^2} + \ln \left( \frac{y}{x} \right) = \ln(x) + C \]

Since we cannot solve for \( y \) this implicit solution is our final answer.

Notice the sum of the exponents in both terms in the numerator and both terms in the denominator is 3. Whenever the sum of the exponents in each term in the problem is the same constant you should consider using the homogeneous substitution to solve the differential equation. The first practice problem should help reinforce this idea.

112.
Solve:
\[ \frac{dy}{dx} = y^4 + x^2y^2 + x^4 \]

113.
Solve:
\[ \frac{dy}{dx} = \frac{2y^5 + x^2y^3 + 3x^4y}{2xy^4 + 3x^5} \]

114.
Solve:
\[ \frac{dy}{dx} = \frac{13y^6 + 18x^2y^4 + 3x^5y}{12xy^5 + 16x^3y^3 + 2x^6} \]

115.
Solve:
\[ \frac{dy}{dx} = \frac{y}{x} + \sqrt{\frac{y}{x}} + 1 \]

116.
Solve:
\[ \frac{dy}{dx} = \frac{6y^3 - 5xy^2 + 2x^2y - x^3}{5xy^2 - 4x^2y + x^3} \]

117.
Solve:
1.4. HOMOGENOUS EQUATION

\[
\frac{dy}{dx} = \frac{y^4 + xy^3 + 2x^2y^2 + x^4}{xy^3}
\]

118. Solve:

\[
\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + \frac{y}{x} + \sqrt{\frac{y}{x}}
\]

119. Solve:

\[
\frac{dy}{dx} = \frac{y^2 \ln \left(\frac{y}{x}\right) + x^2}{xy \ln \left(\frac{y}{x}\right)}
\]

120. Solve:

\[
x \sin \left(\frac{y}{x}\right) \frac{dy}{dx} = y \sin \left(\frac{y}{x}\right) + x
\]

121. Solve:

\[
\frac{dy}{dx} = \frac{y}{x} + \tan^4 \left(\frac{y}{x}\right)
\]

122. Solve:

\[
\frac{dy}{dx} = \left(\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x}}\right)^2
\]

123. Solve:

\[
\left( x - y \arctan \left(\frac{y}{x}\right) \right) dx + x \arctan \left(\frac{y}{x}\right) dy = 0
\]

124. Solve:

\[
\frac{dy}{dx} = \frac{y^2 \tan^2 \left(\frac{y}{x}\right) + x^2}{xy \tan^2 \left(\frac{y}{x}\right)}
\]

125. Solve:

\[
\frac{dy}{dx} = \frac{2y^2 + 2xy + 2x^2}{xy}
\]
126. Solve:

\[ \frac{dy}{dx} = \frac{y^3 + 2xy^2 + x^2y + x^4}{x(x+y)^2} \]

127. Solve:

\[ \frac{dy}{dx} = \frac{y(\ln(y) - \ln(x) + 1)}{x} \]

128. Solve:

\[ x \frac{dy}{dx} = y + \sqrt{x^2 + y^2} \quad y(1) = 0 \]

129. Solve:

\[ \frac{dy}{dx} = \csc^2 \left( \frac{y}{x} \right) + \frac{y}{x} \]

130. Solve:

\[ \frac{dy}{dx} = \frac{2x\ln(e^{\frac{y}{x}})^2}{y^2 + y^2e^{\left(\frac{y}{x}\right)^2} + 2x^2e^{\left(\frac{y}{x}\right)^2}} \]

131. Solve:

\[ \frac{dy}{dx} = \frac{y}{x + \sqrt{xy}} \]

132. Solve:

\[(\sqrt{x+y} + \sqrt{x-y})dx + (\sqrt{x-y} - \sqrt{x+y})dy = 0 \quad y(1) = 1 \]

133. Solve:

\[(2(y-1)(x+y-1) + (y-1)^2)dx + (4(y-1)(x+y-1) + (x+y-1)^2)dy = 0\]

134. Solve:

\[ e^{\frac{y}{x}}(y-x)\frac{dy}{dx} + y(1 + e^{\frac{y}{x}}) = 0 \]

135. Use the substitution \( z = \frac{y}{x} \) to solve the following differential equation
1.5. SHIFT TO HOMOGENOUS

\[ xdy = (x^2 + y^2 + y)dx \]

136. Use the substitution \( z = \frac{y}{x^2} \) to solve the following differential equation

\[ \frac{dy}{dx} = \frac{2y}{x} + \cos \left( \frac{y}{x^2} \right) \]

137. Use the substitution \( z = \frac{y}{x^2} \) to solve the following differential equation

\[ \frac{dy}{dx} = x \left( 1 + \frac{2y}{x^2} + \frac{y^2}{x^4} \right) \]

138. Use the substitution \( u = x^2 + y^2 \) \( v = xy \) to solve the following differential equation

\[ (x^2 + y^2)(xdy + ydx) - xy(xdx + ydy) = 0 \]

139. Use the substitution \( u = x^3 \) \( v = y^2 \) to solve the following differential equation

\[ 3x^5 - y(y^2 - x^3) \frac{dy}{dx} = 0 \]

140. Use the substitution \( u = xy \) to solve the following differential equation

\[ xdy + ydx = x^2y^2dx \]

1.5 Shift to Homogenous

A differential equation can be shifted to produce a Homogenous differential equation if it is of the form:

\[ \frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \]

If the constant terms \( c_1 \) and \( c_2 \) are both zero then the above equation is already Homogenous. This provides the motivation of shifting \( x \) and \( y \) by a constant so that the resulting equation has constant terms \( c_1 \) and \( c_2 \) both equal to zero. To shift this into a Homogenous differential equation use the substitutions:

\[ x = \bar{x} + d_1 \quad y = \bar{y} + d_2 \]

With \( d_1 \) and \( d_2 \) chosen so that the two equations:
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\[ a_1 x + b_1 y + c_1 = 0 \quad a_2 x + b_2 y + c_2 = 0 \]

reduce to:

\[ a_1 \bar{x} + b_1 \bar{y} = 0 \quad a_2 \bar{x} + b_2 \bar{y} = 0 \]

The idea is motivated by the fact that if the constants \( c_1 \) and \( c_2 \) are both zero then:

\[
\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}
\]

is Homogenous.

So choose \( d_1 \) and \( d_2 \) to be the solutions to:

\[
a_1 d_1 + b_1 d_2 + c_1 = 0 \quad a_2 d_1 + b_2 d_2 + c_2 = 0
\]

An Example: Solve:

\[
\frac{dy}{dx} = x - 2y - 2 \quad 2x + y + 6 = 0
\]

Solving system of equations:

\[
x - 2y - 2 = 0 \quad 2x + y + 6 = 0
\]

\[
x = 2 + 2y \quad 2(2 + 2y) + y = -6 \quad 5y = -10 \quad y = -2 \quad x = -2
\]

We will use the substitution \( x = \pi - 2 \) and \( y = \frac{2}{3} - 2 \) making \( \frac{dx}{dy} = \frac{\frac{dy}{d\pi}}{x} \)

Our differential equation becomes:

\[
\frac{d\bar{y}}{d\bar{x}} = \frac{\pi - 2\bar{y}}{2\pi + \bar{y}}
\]

Multiplying the numerator and denominator on the right hand side by \( \frac{1}{\bar{x}} \) gives:

\[
\frac{d\bar{y}}{d\bar{x}} = \frac{1 - 2\bar{y}}{2 + \frac{\bar{y}}{\bar{x}}}
\]

Let \( z = \frac{\bar{y}}{\bar{x}} \quad \bar{y} = z\bar{x} \quad \frac{d\bar{y}}{d\bar{x}} = z + \bar{x} \frac{dz}{d\bar{x}} \)

Under this substitution our differential equation becomes:

\[
z + \bar{x} \frac{dz}{d\bar{x}} = \frac{1 - 2z}{2 + z}
\]

\[
\bar{x} \frac{dz}{d\bar{x}} = \frac{-z^2 - 4z + 1}{z + 2}
\]

\[
\frac{z + 2}{-z^2 - 4z + 1} \frac{dz}{d\bar{x}} = \frac{d\bar{x}}{\bar{x}}
\]

Integrating gives:
\[ -\frac{1}{2} \ln |z^2 - 4z + 1| = \ln |x| + C \]

Multiplying by \(-2\) and negating the contents inside the absolute value bars on the left gives

\[ \ln |z^2 + 4z - 1| = \ln |x|^{-2} + C \]

\[ z^2 + 4z - 1 = \frac{C}{x^2} \]

Completing the square on the left gives

\[ (z + 2)^2 = \frac{C + 5x^2}{x^2} \]

Extracting a square root gives

\[ |z + 2| = \sqrt{\frac{C + 5x^2}{x^2}} \]

Since both \(x\) and \(y\) are greater than \(-2\), \(x > 0\) and \(z + 2 > 0\). So we get

\[ z + 2 = \frac{\sqrt{C + 5x^2}}{x} \]

\[ z = \frac{\sqrt{C + 5x^2}}{x} - \frac{2x}{x} \]

\[ \frac{\bar{y}}{x} = \frac{\sqrt{C + 5x^2}}{x} - \frac{2x}{x} \]

\[ \bar{y} = \sqrt{C + 5x^2} - 2x \]

\[ y + 2 = \sqrt{C + 5(x + 2)^2} - 2(x + 2) \]

and our final solution is

\[ y = \sqrt{C + 5(x + 2)^2} - 2x - 6 \]

The same differential equation can be solved using the substitution \(u = x - 2y - 2\) and \(v = 2x + y + 6\) but the solution is a bit longer (try it and see which solution you prefer).

141.
Solve:

\[ \frac{dy}{dx} = \frac{-8x + 3y + 2}{-9x + 5y - 1} \]

142.
Solve:
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\[ \frac{dy}{dx} = \frac{x - y - 3}{x + y - 1} \]

143. Solve:

\[ \frac{dy}{dx} = \frac{x + 3y + 4}{4y - 3x + 1} \]

144. Solve:

\[ \frac{dy}{dx} = \frac{x - y + 1}{x + y} \]

145. Solve:

\[ \frac{dy}{dx} = \frac{x + y + 3}{2y - x + 3} \]

146. Solve:

\[ \frac{dy}{dx} = \frac{2x + y}{-y - x + 1} \]

147. Solve:

\[ \frac{dy}{dx} = \frac{x + y + 1}{x - y + 3} \]

148. Solve:

\[ \frac{dy}{dx} = \frac{3 - 2x - y}{x + y - 1} \]

149. Solve:

\[ \frac{dy}{dx} = \frac{5x + 4y - 3}{6x - y + 8} \]

150. Solve:

\[ \frac{dy}{dx} = \frac{1}{2} \frac{(x + y - 1)^2}{(x + 2)^2} \]
1.6 The $z^\alpha$ Substitution

In the previous section we learned how to shift an equation into a homogenous equation but our shifting method only worked for equations of the form:

$$\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$$

where the exponents of the $x$ and $y$ term were 1. Now we will study a substitution that works when the exponents are not 1.

**An Example:** Solve:

$$(x^2 y^2 - 2)dy + (xy^3)dx = 0$$

This differential equation is almost homogenous: the sum of the exponents in $x^2 y^2$ and $xy^3$ are both 4. In fact, if the $-2$ was not involved in the equation it would be homogenous. This is a good candidate for a $z^\alpha$ substitution.

$$y = z^\alpha \quad dy = \alpha z^{\alpha - 1} dz$$

Under this substitution the differential equation becomes:

$$(x^2 z^{2\alpha} - 2)\alpha z^{\alpha - 1} dz + xz^{3\alpha} dx = 0$$

$$\alpha(x^2 z^{3\alpha - 1} - 2z^{\alpha - 1})dz + xz^{3\alpha} dx = 0$$

We will now choose $\alpha$ so that the exponents on each term sum to the same value. So we choose $\alpha$ so that:

$$3\alpha + 1 = \alpha - 1 \quad \alpha = -1$$

If $\alpha = -1$ our equation becomes:

$$-(x^2 z^{-4} - 2z^{-2})dz + xz^{-3} dx = 0$$

$$-(x^2 z^{-4} - 2z^{-2}) \frac{dz}{dx} + xz^{-3} = 0$$

$$\frac{dz}{dx} = \frac{xz^{-3}}{x^2 z^{-4} - 2z^{-2}}$$

$$\frac{dz}{dx} = \frac{xz}{x^2 - 2z^2}$$

Dividing both numerator and denominator by $x^2$ gives

$$\frac{dz}{dx} = \frac{z}{1 - 2z^2}$$

This is now a homogenous equation so we make the following substitution:

$$u = \frac{z}{x} \quad z = ux \quad \frac{dz}{dx} = u + x \frac{du}{dx}$$
Under this substitution our equation becomes:

\[ u + x \frac{du}{dx} = \frac{u}{1 - 2u^2} \]

This is now a separable differential equation

\[ x \frac{du}{dx} = \frac{2u^3}{1 - 2u^2} \]

\[ \frac{1 - 2u^2}{2u^3} du = \frac{dx}{x} \]

Integrating gives

\[ -\frac{1}{4u^2} - \ln|u| = \ln|x| + C \]

Since \( u = \frac{z}{x} \) and \( z = y^{-1} \ u = \frac{1}{xy} \) the solution becomes

\[ -\frac{x^2 y^2}{4} - \ln\left|\frac{1}{xy}\right| = \ln|x| + C \]

Or

\[ -\frac{x^2 y^2}{4} + \ln|xy| = \ln|x| + C \]

151.

Solve:

\[ (2x^2y - 1)dy + 2xy^2 dx = 0 \]

152.

Solve:

\[ (x^2y + 1)dy + xy^2 dx = 0 \]

153.

Solve:

\[ (xy^3 + 1)dy + y^4 dx = 0 \]

154.

Solve:

\[ (x^2y^4 + 4)dy + xy^5 dx = 0 \]

155.

Solve:

\[ (xy + 2)dy - y^2 dx = 0 \]
1.7 Equations of the form: $y' = G(ax + by + c)$

If a differential equation is of the form:

$$\frac{dy}{dx} = G(ax + by + c)$$

Then use the following substitution to transform the equation into a Separable Differential Equation:

$$z = ax + by + c \quad \quad \frac{dy}{dx} = \frac{1}{b} \left( \frac{dz}{dx} - a \right)$$

Under this substitution our differential equation becomes

$$\frac{1}{b} \left( \frac{dz}{dx} - a \right) = G(z)$$

which reduces to the Separable Differential Equation:

$$\frac{dz}{a + bG(z)} = dx$$

An Example: Solve:

$$\frac{dy}{dx} = \left( x + y \right) \left( 1 + \ln(x + y) \right) - 1$$

Let $z = x + y \quad \quad \frac{dy}{dx} = \frac{dz}{dx} - 1$

Under this substitution our differential equation becomes:

$$\frac{dz}{dx} - 1 = z(1 + \ln(z)) - 1$$

$$\int \frac{dz}{z(1 + \ln(z))} = \int dx$$

$$\ln(1 + \ln(z)) = x + C$$

$$1 + \ln(x + y) = Ce^x$$

$$\ln(x + y) = Ce^x - 1$$

$$x + y = e^{Ce^x - 1}$$

And our final solution is:

$$y = e^{Ce^x - 1} - x$$
CHAPTER 1. FIRST ORDER EQUATIONS.

157.
Solve:

\[ \frac{dy}{dx} = (x + y - 4)^2 \]

158.
Solve:

\[ 3 \frac{dy}{dx} = (2x + 3y - 1) + 4(2x + 3y - 1)^{-3} - 2 \]

159.
Solve:

\[ 2 \frac{dy}{dx} = \frac{1}{(x + 2y + 1)e^{(x+2y+1)x} - 1} \]

160.
Solve:

\[ \frac{dy}{dx} = \tan^2(x + y) \]

161.
Solve:

\[ \frac{dy}{dx} = \sin^2(y - x) \]

162.
Solve:

\[ \frac{dy}{dx} = \csc^2(4x + y + 1) - 4 \]

163.
Solve:

\[ \frac{dy}{dx} = \sin(x + y) \]

164.
Solve:

\[ \frac{dy}{dx} = \frac{1}{\sqrt{x + y}} \]

165.
Solve:

\[ \frac{dy}{dx} = \sqrt{e^{2x+2y} - 1} - 1 \]

166.
Solve:

\[ 2 \frac{dy}{dx} = \sec(2y - 4x + 1) + \tan(2y - 4x + 1) + 4 \]
1.7. EQUATIONS OF THE FORM: \( y' = g(ax + by + c) \)

Solve:

\[
\frac{dy}{dx} = \frac{x - y + \sqrt{1 + (x - y)^2}}{\sqrt{1 + (x - y)^2}}
\]

167.

Solve:

\[
\frac{dy}{dx} = \frac{4(x - y) \ln(x - y) - 1}{4(x - y) \ln(x - y)}
\]

168.

Solve:

\[
\frac{dy}{dx} = 1 + \sqrt{e^{2y - 2x} - 1}
\]

169.

Solve:

\[
\frac{dy}{dx} = \frac{2e^{-x-y}}{e^x + y - e^{-x-y}}
\]

170.

Solve:

\[
\frac{dy}{dx} = (4 + (4x + y)^2)^{3/2} - 4
\]

171.

Solve:

\[
e^{-y} \left( \frac{dy}{dx} + 1 \right) = xe^x
\]

172.

Solve:

\[
\frac{dy}{dx} = \left( \cos^3(x + y) - 1 \right) \left( \cos^3(x + y) + 1 \right)
\]

173.

Solve:

\[
\frac{dy}{dx} = \sin(2x + 2y) - \sin^2(x + y)
\]

174.

Use the substitution \( z = y + x \) to solve

\[
\frac{dy}{dx} + \frac{2y}{x} + 3 = x^2(x + y)^3
\]

175.

Use the substitution \( z = y + x \) to solve

\[
\frac{dy}{dx} + \frac{2x + y}{x} = \frac{4x}{x + y}
\]
Use the substitution \( z = y - x \) to solve

\[
\frac{dy}{dx} + x(y - x) + x^3(y - x)^2 = 1
\]

### 1.8 Exact Equations

In Multivariable we analyzed the functions of the form \( z = f(x, y) \) by studying their level sets. The level sets of this three dimensional function can be graphed in two dimensional space by replacing \( z \) with different constants and analyzing the resulting two dimensional graphs. The total differential of:

\[
f(x, y) = C \quad \text{is} \quad f_x dx + f_y dy = 0
\]

We also know that mixed partials are equal meaning:

\[
f_{xy} = f_{yx} \quad \text{for all} \quad f(x, y) = C
\]

We will now study the Exact Differential Equation. An Exact Differential Equation is an equation that was created by calculating the total differential of some function of two or more variable set equal to a constant. In calculus we learned to go from

\[
f(x, y) = C \quad \text{to} \quad f_x dx + f_y dy = 0
\]

In differential equations we will learn to go from

\[
f_x dx + f_y dy = 0 \quad \text{to} \quad f(x, y) = C
\]

The equation

\[
M dx + N dy = 0 \quad \text{is Exact if} \quad M_y = N_x
\]

If an equation is exact then \( M \) is the partial derivative of some function \( f(x, y) = C \) with respect to \( x \) and \( N \) is the partial derivative of the same function with respect to \( y \). So to find \( f \) up to a constant we need to integrate \( M \) with respect to \( x \) (obtaining a constant which will be a function \( g(y) \)) or we need to integrate \( N \) with respect to \( y \) (obtaining a constant which will be a function \( g(x) \)). That is:

\[
f(x, y) = \int M dx + g(y)
\]

We then solve for \( g(y) \) using the fact that \( f_y = N \). That is:

\[
f_y = \frac{\partial}{\partial y} \int M dx + g'(y) = N
\]

And the solve for \( g(y) \) and insert it into our formula for \( f(x, y) \) and write the final answer as a level set \( f(x, y) = C \)

**An Example**: Solve:
(8xy^3 + 2xy + 3x^2)dx + (12x^2y^2 + x^2 + 4y^3)dy = 0

Solution:

\[ M = 8xy^3 + 2xy + 3x^2 \quad N = 12x^2y^2 + x^2 + 4y^3 \]

Test for exactness:

\[ M_y = 24xy^2 + 2x \quad N_x = 24xy^2 + 2x \]

Since \( M_y = N_x \), our differential equation is exact, making \( M = 8xy^3 + 2xy + 3x^2 \) the partial derivative of the function \( F \) we are solving for with respect to \( x \). That is:

\[ F_x = 8xy^3 + 2xy + 3x^2 \]

So

\[ F = \int (8xy^3 + 2xy + 3x^2)dx = 4x^2y^3 + x^2y + x^3 + g(y) \]

Since this equation is exact \( F_y = N \). This creates the equation:

\[ F_y = 12x^2y^2 + x^2 + g'(y) = 12x^2y^2 + x^2 + 4y^3 \]

So

\[ g'(y) = 4y^3 \quad g(y) = y^4 \]

Our final solution is:

\[ 4x^2y^3 + x^2y + x^3 + y^4 = C \]

177. Solve:

\[ \left( 6xy + 6y^4 - 24x^2 \right)dx + \left( 3x^2 + 24xy^3 + \frac{6}{y} \right)dy = 0 \]

178. Solve:

\[ \left( \frac{-y}{x^2 + y^2} \right)dx + \left( \frac{x}{x^2 + y^2} \right)dy = 0 \quad y(1) = 1 \]

179. Solve:

\[ \left( y \cos(y) + y \sin(x) + xy \cos(x) \right)dx + \left( x \sin(x) + x \cos(y) - xy \sin(y) \right)dy = 0 \]

180. Solve:
\[
\left( \sin(y) + y \sin(x) + 2e^{2x} \right) dx + \left( x \cos(y) - \cos(x) + \sin(y) \right) dy = 0
\]

181. Solve:

\[
(2xe^{xy} + x^2 ye^{xy}) dx + (x^3 e^{xy}) dy = 0 \quad y(1) = \ln(4)
\]

182. Solve:

\[
\left( 3x^2 \sin(y) + y \cos(y) \right) dx + \left( x^3 \cos(y) + x \cos(y) - xy \sin(y) \right) dy = 0
\]

183. Solve:

\[
\left( xy^2 \cos(xy) + 1 \right) dx + \left( x^2 y \cos(xy) + x \sin(xy) \right) dy = 0
\]

184. Solve:

\[
\left( xy(e^{xy} + 1) + y \right) dx + \left( x^2 e^{xy} + x \right) dy = 0
\]

185. Solve:

\[
\left( xe^x - e^x + y \right) dx + \left( 1 + \ln(y) + x \right) dy = 0
\]

186. Solve:

\[
\left( \frac{y(y^2 - x^2)}{(x^2 + y^2)^2} \right) dx + \left( \frac{x(x^2 - y^2)}{(x^2 + y^2)^2} \right) dy = 0
\]

187. Find the function \( M(x, y) \) that makes the following differential equation exact.

\[
M(x, y) dx + \left( xe^{xy} + 2xy + \frac{1}{x} \right) dy = 0
\]

188. The following differential equation arises from the total differential of a function \( F \) with variables: \( x, y \) and \( z \) set equal to a constant. Find this function \( F(x, y, z) \)

\[
(2yz^2 + 6xz) dx + (2xz^2 + 30y^2 z) dy + (4xyz + 3x^2 + 10y^3) dz = 0
\]

189. The following differential equation arises from the total differential of a function \( F \) with variables: \( x, y \) and \( z \) set equal to a constant. Find this function \( F(x, y, z) \)

\[
(yz + 1) dx + (xz + 2y) dy + (xy + 3z^2) dz = 0
\]
1.9 Integrating Factors for non-exact Equations

If a differential equation is not exact: \( M_y \neq N_x \) sometimes we can multiply both sides of the equation by an integrating factor \( I \) to make it exact. We need a formula for this integrating factor. Starting with:

\[
M \, dx + N \, dy = 0 \quad \text{and multiplying both sides by } I \text{ gives } IM \, dx + IN \, dy = 0
\]

For this to be exact \( \frac{\partial}{\partial y} (IM) = \frac{\partial}{\partial x} (IN) \). Calculating these partials gives:

\[
I_y M + M_y I = I_x N + N_x I
\]

This is a partial differential equation that we cannot solve. So we will solve a special case: the case where \( I \) is a function of one variable. Case 1: \( I \) is a function of \( x \) making \( I_y = 0 \). Now our partial differential equation is a bit easier to solve:

\[
M_y I = I_x N + N_x I
\]

Or

\[
I(M_y - N_x) = I_x N
\]

Which becomes:

\[
\frac{I_x}{I} = \frac{M_y - N_x}{N}
\]

Integrating both sides with respect to \( x \) gives:

\[
\ln |I| = \int \left( \frac{M_y - N_x}{N} \right) \, dx
\]

Making our integrating factor:

\[
I(x) = e^{\int \left( \frac{M_y - N_x}{N} \right) \, dx}
\]

Case 2: \( I \) is a function of \( y \) making \( I_x = 0 \). In this case the integrating factor is:

\[
I(y) = e^{\int \left( \frac{N_x - M_y}{-M} \right) \, dy}
\]

An Example: Solve:

\[
(2y^7 + y^4) \, dx + (6xy^6 - 3) \, dy = 0
\]

\[
M = 2y^7 + y^4 \quad N = 6xy^6 - 3
\]

Test for exactness:

\[
M_y = 14y^6 + 4y^3 \quad N_x = 6y^6
\]
Since \( M_y \neq N_x \) our equation is not exact. We will look for an integrating factor. Since

\[
\frac{N_x - M_y}{M} = \frac{6y^6 - 14y^6 - 4y^3}{2y^7 + y^4} = \frac{-4y^3(2y^3 + 1)}{y^4(2y^3 + 1)} = -\frac{4}{y}
\]

is a function of only \( y \) our integrating factor is:

\[
I = e^{\int \frac{N_x - M_y}{M} \, dy} = e^{\int \frac{-4}{y} \, dy} = y^{-4}
\]

Multiplying both sides of the differential equation by \( y^{-4} \) gives:

\[
(2y^3 + 1)dx + (6xy^2 - 3y^{-4})dy = 0
\]

\[
M = 2y^3 + 1 \quad N = 6xy^2 - 3y^{-4}
\]

Now

\[
M_y = 6y^2 = N_x
\]

Since \( M_y = N_x \) our differential equation is exact, making \( M = 2y^3 + 1 \) the partial derivative of the function \( F \) we are solving for with respect to \( x \). That is:

\[
F_x = 2y^3 + 1
\]

So

\[
F = \int (2y^3 + 1) \, dx = 2xy^3 + x + g(y)
\]

And \( F_y = N \)

\[
F_y = 6xy^2 + g'(y) = 6xy^2 - 3y^{-4} \quad g'(y) = -3y^{-4} \quad g(y) = y^{-3}
\]

And our final solution is:

\[
F(x, y) = 2xy^3 + x + y^{-3} = C
\]

190. Solve:

\[
\left( 3y + 5x^2y^3 + 4x \right) dx + \left( x + 3x^3y^2 \right) dy = 0
\]

191. Solve:

\[
\left( 2x^4 + x^3ye^{xy} - 2 \right) dx + \left( 2x^3y + x^4e^{xy} \right) dy = 0
\]

192. Solve:
(2x^2 + y)dx + (x^2y - x)dy = 0 \quad y(1) = 1

193. Solve:

(16x^2y^2 + 64x^4 + 1)dx + (4xy^3 + 16x^3y) = 0 \quad y(1) = 1

194. Solve:

(xe^y)dx + (ye^{y^2})dy = 0 \quad y(0) = 0

195. Solve:

(2xy^7 + y)dx + (3x^2y^6 - 3x)dy = 0 \quad y(1) = 1

196. Solve:

\left(2e^{2x} + e^y\right)dx + \left(3e^{2x} + 4xe^y\right)dy = 0

197. Solve:

\left(3 + \frac{6xy}{1 + x^2}\right)dx + 3dy = 0

198. Solve:

\left(e^{x^2}(2x^2 + 1) + e^{-y^2}(2y^3 + 1)\right)dx + \left(2xye^{x^2} + e^{-y^2}(6xy^2 + 1)\right)dy = 0

199. Solve:

3xydx + (x^2 + 1)dy = 0

Sometimes we cannot find an integrating factor of just a single variable using the above formulas. In this case we guess the form of the integrating factor and try to find constants that make it work.

200. Solve by finding an integrating factor of the form \( I = x^n y^m \):

\left(3y^4 + 18y^{-1}\right)dx + \left(5xy^3 + 2x^{-2}\right)dy = 0

201. Solve by finding an integrating factor of the form \( I = x^n y^m \):
\[
\frac{y^3}{x(x+y)^2} \, dx + \frac{xy}{(x+y)^2} \, dy = 0
\]

202.
Solve by finding an integrating factor of the form \( I = x^n y^m \):
\[
2 \cos(x^2 y^2)(y \, dx + x \, dy) = 0
\]

203.
If \( I(x) = x \) is an integrating factor for
\[
f(x) \frac{dy}{dx} + x^2 + y = 0
\]
Find all functions \( f(x) \)

204.
Solve by finding an integrating factor of the form \( I = e^{kx} \cos(y) \):
\[
\frac{dy}{dx} = \tan(y) - e^x \sec(y)
\]

205.
Solve by finding an integrating factor of the form \( I = \sin^n(x) \cos^m(y) \):
\[
\left(4 \cos(x) + 3 \cot(x)\right) \, dx + \left(-2 \sin(x) \tan(y) - 3 \tan(y)\right) \, dy = 0
\]

206.
Solve:
Show that \( I = \frac{1}{x^2+y^2} \) is an integrating factor for
\[
\left(y + xf(x^2 + y^2)\right) \, dx + \left(yf(x^2 + y^2) - x\right) \, dy = 0
\]
Use this result to solve
\[
\left(y + x(x^2 + y^2)^2\right) \, dx + \left(y(x^2 + y^2)^2 - x\right) \, dy = 0
\]

207.
Show that
\[
I = \frac{1}{Ax^2 + Bxy + Cy^2}
\]
is an integrating factor for
\[
xdy - ydx = 0
\]

208.
Show that
\[
G(x, y) = \ln(x+y) - \frac{1}{x+y}
\]
is a solution to the exact equation

\[
\frac{1}{x+y} + \frac{1}{(x+y)^2} \, dx + \left( \frac{1}{x+y} + \frac{1}{(x+y)^2} \right) \, dy = 0
\]

Now multiply both sides of this exact equation by \((x+y)^2\) producing

\[(x+y+1) \, dx + (x+y+1) \, dy = 0\]

which has a solution

\[F(x,y) = \frac{1}{2} x^2 + \frac{1}{2} y^2 + xy + x + y\]

What is the relationship between \(F\) and \(G\)?

Now show in general that if \(F(x,y) = C\) is a solution to

\[F_x \, dx + F_y \, dy = 0\]

and \(G(x,y) = C\) is a solution to

\[I(x,y)F_x \, dx + I(x,y)F_y \, dy = 0\]

then

\[F_x G_y = G_x F_y\]

209.

In the study of first order linear differential equations:

\[\frac{dy}{dx} + P(x) y = Q(x)\]

we learned that multiplying by the integrating factor:

\[I(x) = e^{\int P(x) \, dx}\]

will transform the equation into a separable differential equation. Show that by multiplying both sides of

\[\left( P(x)y - Q(x) \right) \, dx + dy = 0\]

by the same integrating factor will transform the above equation into an exact equation.

**Bernoulli’s Spread of Smallpox:**

Let \(x(t)\) represent the population of all living people.

Let \(y(t)\) represent the population of all living people who have not contracted smallpox.

Let \(a > 0\) be the rate at which population \(y(t)\) contracts smallpox.

Therefore

\[\frac{dy}{dt} = -ay\]

Let \(0 < b < 1\) be the precentage of population \(y(t)\) that get and die from smallpox.
Therefore

\[ \frac{dx}{dt} = -aby \]

Let \( d(t) \) be the average death rate of populations \( x(t) \) and \( y(t) \) from causes other than smallpox. Without smallpox we have

\[ \frac{dx}{dt} = -d(t)x(t) \quad \frac{dy}{dt} = -d(t)y(t) \]

With smallpox we have

\[ \frac{dx}{dt} = -d(t)x(t) - aby(t) \quad \frac{dy}{dt} = -d(t)y(t) - ay(t) \]

To eliminate the average death rate \( d(t) \) we multiply \( \frac{dx}{dt} \) by \( y(t) \) and \( \frac{dy}{dt} \) by \( x(t) \) and subtract

\[ y \frac{dx}{dt} - x \frac{dy}{dt} = axy - aby^2 \]

Recognizing the left hand is the numerator of \( \frac{dx}{dt} \) we now divide both sides by \( y^2 \)

\[ \frac{y \frac{dx}{dt} - x \frac{dy}{dt}}{y^2} = \frac{ax}{y} - ab \]

This equation is now

\[ \frac{d}{dt} y \frac{dx}{dt} = \frac{ax}{y} - ab \]

The substitution \( z = \frac{x}{y} \) yields the separable equation

\[ \frac{dz}{dt} = az - ab \]

\[ \frac{dz}{z - b} = adt \]

\[ \ln(z - b) = at + C \]

\[ z - b = Ce^{at} \]

\[ y = x(Ce^{at} + b) \]

### 1.10 Orthogonal Trajectories

A common geometric problem in many applications involves finding a family of curves: **Orthogonal Trajectories**, that intersect a given family of curves orthogonally at each point. If you are given a family of curves in the form \( F(x, y) = K \) then the slope of this family of curves is given by the derivative:

\[ \frac{dy}{dx} = -\frac{F_x}{F_y} \]
1.10. ORTHOGONAL TRAJECTORIES

Since we are looking for an orthogonal family of curves, and in $R^2$ orthogonal lines have negative reciprocal slopes, we will be for a family of curves that satisfy the following differential equation:

$$y' = \frac{F_y}{F_x}$$

_An Example:_

Find the orthogonal trajectories for the circle:

$$x^2 + y^2 = r^2$$

To find the orthogonal trajectories we take $F(x, y) = x^2 + y^2$ we will need to solve the differential equation:

$$y' = \frac{2y}{2x} = \frac{y}{x}$$

This equation is separable

$$\frac{dy}{y} = \frac{dx}{x}$$

So

$$\ln(y) = \ln(x) + C$$

The family of curves orthogonal to the circle is:

$$y = kx$$

Conversely, the family of curves orthogonal to the lines $y = kx$ is given by the circle $x^2 + y^2 = r^2$.

_An Example:_ Find the orthogonal trajectories for the circle:

$$x^2 + y^2 = Cx$$

Here we will take $F(x, y) = x^2 + y^2 - Cx$ and we are looking for a family of curves that satisfy the following:

$$y' = \frac{2y}{2x - C}$$

The problem with this differential equation is that it involves the constant $C$. To eliminate this constant we will solve for $C$ in the original equation of the circle and insert it into the differential equation.

$$C = \frac{x^2 + y^2}{x}$$

Our differential equation becomes:

$$y' = \frac{2xy}{x^2 - y^2}$$
Recognizing that the sum of the exponents in each term in both the numerator and denominator add to the same value of 2 we see that this is a homogenous equation. Remember, to solve the homogenous equation you must first write it as:

\[
\frac{dy}{dx} = f\left(\frac{y}{x}\right)
\]

and make the substitution

\[z = \frac{y}{x}\]

Multiplying both the numerator and denominator by \(\frac{1}{x^2}\) we get the homogenous equation:

\[
\frac{dy}{dx} = \frac{2\frac{y}{x}}{1 - \left(\frac{y}{x}\right)^2}
\]

Making the substitutions

\[z = \frac{y}{x} \quad \frac{dy}{dx} = x \frac{dz}{dx} + z\]

Our differential equation becomes:

\[
x \frac{dz}{dx} + z = \frac{2z}{1 - z^2}
\]

This equation is now separable:

\[
x \frac{dz}{dx} = \frac{z^3 + z}{1 - z^2}
\]

\[
\int \frac{1 - z^2}{z(z^2 + 1)} dz = \int \frac{dx}{x}
\]

The integral on the left requires partial fraction decomposition. After applying partial fraction we get:

\[
\int \left(\frac{1}{z} - \frac{2z}{1 + z^2}\right) dz = \int \frac{dx}{x}
\]

Integrating

\[
\ln(z) - \ln(1 + z^2) = \ln(x) + C
\]

\[
\ln\left(\frac{z}{1 + z^2}\right) = \ln(x) + C
\]

\[
\frac{z}{1 + z^2} = Cx
\]

\[
z = Cx + Cxz^2
\]

Expressing this quadratic with the coefficient of \(z^2\) being 1 gives

\[
z^2 - \frac{1}{Cx}z + 1 = 0
\]
To solve for $z$ in this equation we must complete the square by adding $\frac{1}{4C^2x^2}$ to both sides:

$$z^2 - \frac{1}{C}z + \frac{1}{4C^2x^2} + 1 = \frac{1}{4C^2x^2}$$

Factoring the first 3 terms on the left we get

$$\left(z - \frac{1}{2C} \right)^2 = \frac{1}{4C^2x^2} - 1$$

$$z - \frac{1}{2C} = \pm \sqrt{\frac{1 - 4C^2x^2}{4C^2x^2}}$$

$$z = \frac{1}{2C} \pm \frac{\sqrt{1 - 4C^2x^2}}{2C}$$

Solving for $y$ remembering that $z = \frac{y}{x}$ gives:

$$y = x \left( \frac{1 \pm \sqrt{1 - 4C^2x^2}}{2C} \right)$$

The orthogonal family of curves we desire is:

$$y = \frac{1}{2C} \pm \frac{\sqrt{1 - 4C^2x^2}}{2C}$$

Sometimes we are concerned with finding a family of curves that make an angle of $\alpha \neq 90^\circ$ with a given family of curves. These curves are called Oblique Trajectories.

Given a family of curves: $F(x, y) = K$ with its derivative (slope) given by:

$$\frac{dy}{dx} = f(x, y)$$

Treating $dy$ as the change in $y$ and $dx$ as the change in $x$ and creating a triangle gives:

From the triangle we see, $\tan(\theta) = \frac{dy}{dx} = f(x, y)$ so the tangent line has an angle of inclination of $\arctan(f(x, y))$. So the tangent line of an oblique trajectory that intersects this curve at an angle of $\alpha$ will have an angle of inclination of:

$$\arctan(f(x, y)) + \alpha$$

Making the slope of the oblique trajectory

$$\tan \left( \arctan(f(x, y)) + \alpha \right) = \frac{f(x, y) + \tan(\alpha)}{1 - f(x, y) \tan(\alpha)}$$
Thus the differential equation of this family of oblique trajectories is given by:

\[ y' = \frac{f(x, y) + \tan(\alpha)}{1 - f(x, y) \tan(\alpha)} \]

An Example:

Find the family of oblique trajectories that intersect the family of straight lines \( y = Cx \) at an angle of 45°. Here we take \( F(x, y) = y - Cx \) and compute the slope of the tangent line:

\[ \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{C}{1} = C \]

Solving for \( C \) we get

\[ \frac{dy}{dx} = \frac{y}{x} = f(x, y) \]

Using this function for \( f(x, y) \) and \( \alpha = 45^\circ \) the differential equation of this family of oblique trajectories is

\[ y' = \frac{\frac{y}{x} + \tan(45^\circ)}{1 - \frac{y}{x} \tan(45^\circ)} = \frac{x + y}{x - y} \]

Multiplying both numerator and denominator by \( \frac{1}{x} \) produces

\[ \frac{dy}{dx} = \frac{1 + \frac{y}{x}}{1 - \frac{y}{x}} \]

This is a homogenous equation so we make the following substitution

\[ z = \frac{y}{x} \quad \frac{dy}{dx} = z + x \frac{dz}{dx} \]

The differential equation becomes

\[ z + x \frac{dz}{dx} = 1 + \frac{z}{1 - z} \]

This equation is now separable

\[ \frac{1 - z}{1 + z^2} \frac{dz}{dx} = \frac{dx}{x} \]

Integrating gives

\[ \arctan(z) - \frac{1}{2} \ln(1 + z^2) = \ln(x) + C \]

Treating \( C \) as \( \ln(K) \) and using some properties of logs we get

\[ 2 \arctan(z) = \ln(K^2 x^2 (1 + z^2)) \]

Converting back to \( x \) and \( y \) we get the family of oblique trajectories

\[ 2 \arctan\left(\frac{y}{x}\right) = \ln(K^2 (x^2 + y^2)) \]
210. Find the orthogonal trajectories for the family of straight lines.

\[ y = mx + 1 \]

211. Find the orthogonal trajectories for each given family of curves.

\[ y = Cx^3 \]

212. Find the orthogonal trajectories for each given family of curves.

\[ x^2 + y^2 = Cx^3 \]

213. Find the orthogonal trajectories for each given family of curves.

\[ y = Ce^{2x} \]

214. Find the orthogonal trajectories for each given family of curves.

\[ y = x - 1 + Ce^{-x} \]

215. Find the orthogonal trajectories for each given family of curves.

\[ x - y = Cx^2 \]

216. Find the value of \( n \) so that the curves \( x^n + y^n = C \) are the orthogonal trajectories of

\[ y = \frac{x}{1 + Kx} \]

217. Show that the following family of curves is self orthogonal.

\[ y^2 = 4C(x + C) \]

218. Show that the following family of curves is self orthogonal.

\[ y^2 = 2Cx + C^2 \]

219. Find a family of oblique trajectories that intersect the family of circles \( x^2 + y^2 = r^2 \) at an angle of 45°. 

220.
Find a family of oblique trajectories that intersect the family of circles $y^2 = Cx^2$ at an angle of $60^\circ$.

Let $O$ be the origin and $P$ be a point on the curve $P(x)$. Let $N$ be the point on the x-axis where the normal line to $P(x)$ intersects the x-axis. If $\overline{OP} = \overline{ON}$ what is the equation of $P(x)$?
Chapter 2

Second Order Equations.

2.1 Wronskian, Fundamental Sets and Able’s Theorem

In this section we will mostly be dealing with the second order linear differential equation:

\[ y'' + P(x)y' + Q(x)y = 0 \]

If we want to find all solutions to this equation it can be shown that we are looking for two solutions \( y_1 \) and \( y_2 \) to the equations with the one restriction that \( y_1 \) and \( y_2 \) cannot be scalar multiples of each other. But in order to expand our knowledge to third order and higher order equations we replace the restriction that the solutions cannot be scalar multiples of each other with the restriction that the set of solutions must be linearly independent.

The Set of functions \( \{y_1, y_2, ..., y_n\} \) is Linearly Independent if the only solution to

\[ C_1y_1 + C_2y_2 + ... + C_ny_n = 0 \]

is \( C_1 = C_2 = ... = C_n = 0 \)

Although this is the formal definition of Linearly Independent sets we will not be using it. Instead we will be using the Wronskian to determine if a set is linearly independent. Since we are considering only second order equations in this chapter we will limit our study to the linear independence or dependence of two functions \( y_1 \) and \( y_2 \)

The Wronskian of two functions \( y_1 \) and \( y_2 \) is given by the determinate:

\[
W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1 y'_2 - y_2 y'_1
\]

If two functions \( y_1 \) and \( y_2 \) are linearly dependent then, from the formal definition, is possible to express one function as a scalar multiple of the other. That is:

\[ y_2 = C y_1 \]

Making the Wronskian: 55
CHAPTER 2. SECOND ORDER EQUATIONS.

\[ W(y_1, y_2) = W(y_1, Cy_1) = \begin{vmatrix} y_1 & Cy_1 \\ y'_1 & Cy'_1 \end{vmatrix} = Cy_1 y'_1 - Cy_1 y'_1 = 0 \]

So if two functions are linearly dependent their Wronskian is identically zero.

If \( y_1 \) and \( y_2 \) are both solutions to \( y'' + P(x)y' + Q(x)y = 0 \) and \( \{y_1, y_2\} \) is linearly independent then \( \{y_1, y_2\} \) is a **Fundamental Solution Set** of the differential equation.

222. Show that the set is a fundamental solution set of the differential equation

\[ \{y_1, y_2\} = \{e^{3x}, xe^{3x}\} \quad y'' - 6y' + 9y = 0 \]

223. Show that the set is a fundamental solution set of the differential equation

\[ \{y_1, y_2\} = \{x^2, x^3\} \quad x^2 y'' - 4xy' + 6y = 0 \]

224. Show that the set is a fundamental solution set of the differential equation

\[ \{y_1, y_2\} = \{\sinh(x), \cosh(x)\} \quad y'' - y = 0 \]

Remember

\[
\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \cosh(x) = \frac{e^x + e^{-x}}{2}
\]

225. Show that the set is a fundamental solution set of the differential equation

\[ \{y_1, y_2, y_3\} = \{e^x, e^{2x}, e^{3x}\} \quad y''' - 6y'' + 11y' - 6y = 0 \]

226. Show that the set is a fundamental solution set of the differential equation

\[ \{y_1, y_2\} = \{e^{ax} \sin(bx), e^{ax} \cos(bx)\} \quad y'' - 2ay' + (a^2 + b^2)y = 0 \]

227. Show that the two sets are both fundamental solution set of the differential equation. Which one would you rather work with?

\[ S_1 = \{y_1, y_2\} = \{e^x, e^{2x}\} \quad S_2 = \{y_1, y_2\} = \{4e^x, e^{2x} - 6e^x\} \quad y'' - 3y' + 2y = 0 \]

228. Show that if \( y_1 \) and \( y_2 \) both have a relative extrema at \( x = x_0 \) then they cannot be a fundamental solution set to \( y'' + P(x)y' + Q(x)y = 0 \) on an interval containing \( x = x_0 \).

229. Show that \( y_1 = \cos(2x) \) and \( y_2 = \cos^2(x) - \sin^2(x) \) are Linearly Dependent
230. Show for a, b Constants that

\[ W(ay_1, by_2) = abW(y_1, y_2) \]

231. Show for f, g, h differentiable functions that

\[ W(fg, fh) = f^2W(g, h) \]

232. Calculate and simplify the following

\[ e^\int W\left(\frac{f}{g}, \frac{f}{g}\right) \, dx \]

233. Show

\[ W(y_1 + a, y_2 + a) = W(y_1, y_2) + a \frac{d}{dx} (y_2 - y_1) \]

234. Show

\[ W(y_1, y_2) = y_2^2 \frac{d}{dx} \left( \frac{y_1}{y_2} \right) \]

235. Show

\[ y_1 \cdot W\left(\frac{y_2}{y_1}, y_1\right) + y_2 \cdot W\left(\frac{y_1}{y_2}, y_2\right) = \frac{d}{dx} (y_1 \cdot y_2) \]

236. If \( W(y_1, y_2) = e^{4x} \) and \( y_1 = e^x \) find \( y_2 \) if \( y_2(0) = 2 \)

237. Show:

\[ W(f, g + h) = W(f, g) + W(f, h) \]

238. If

\[ W(f, g) = e^{5x} \]

Find

\[ W(f + g, f - g) \]

239. If

\[ W(y, y^2) = e^{3x} \]

\[ y(0) = 1 \]
Find \( y(x) \)

240.

Show

\[ W'(f, g) = W(f, g') + W(f', g) \]

241.

If \( W(f(x), g(x))|_{x=0} = 10 \) Find \( W(f(3x), g(3x))|_{x=0} \)

242.

If

\[ W(f, g) = W(g, h) \]

Show

\[ g(x) = C(f(x) + h(x)) \]

243.

If

\[ W(f, g) = f \cdot g \]

Find

\[ \frac{g}{f} \]

244.

Show:

\[ W\left(f(x) \cos(x), f(x) \sin(x)\right) = \left(f(x)\right)^2 \]

245.

If

\[ W(f(g), g) = 0 \]

Show

\[ f(x) = Cx \]

Interpret the results in terms of linear independence

246.

Solve for \( f(x) \) if

\[ W(x, f) = \frac{\sqrt{1 + x^2}}{x^2} \]

247.

Find
2.1. WRONSKIAN, FUNDAMENTAL SETS AND ABLE’S THEOREM

\[ \int f W(f, \frac{d}{dx}) \, dx \]

248.
Show:

\[ W(x \cdot f, f) \leq 0 \]

249.
Let \( y_1 \) and \( y_2 \) be solutions to

\[ y'' + P(x)y' + Q(x)y = 0 \]

Show:

\[ P(x) = -\frac{y_1y_2'' - y_2y_1''}{W(y_1, y_2)} \quad \text{and} \quad Q(x) = \frac{y_1'y_2'' - y_2'y_1''}{W(y_1, y_2)} \]

Use this result to show the differential equation with solutions \( y_1 = x^n \) and \( y_2 = x^m \) with \( n \neq m \) has the form

\[ ax^2 y'' + bxy' + cy = 0 \]

250.
Show the second order linear homogenous equation

\[ y'' + P(x)y' + Q(x)y = 0 \]

with fundamental solution set \( \{ y_1, y_2 \} \) and Wronskian \( W \) can be written as

\[ \frac{1}{W} \begin{vmatrix} y & y_1 & y_2 \\ y' & y_1' & y_2' \\ y'' & y_1'' & y_2'' \end{vmatrix} = 0 \]

251.
Use the results of either of the previous problem to find a differential equation with the following Fundamental Solution Set

\[ \{ y_1, y_2 \} = \{ \sin(kx), \cos(kx) \} \]

252.
Show

\[ W \left( 1, x, x^2, x^3, ..., x^n \right) = 1! \cdot 2! \cdot 3! \cdot ... \cdot n! \]

253.
The differential equation

\[ y'' + P(x)y' + Q(x)y = 0 \]

can be converted into Normal Form.
CHAPTER 2. SECOND ORDER EQUATIONS.

\[ u'' + f(x)u = 0 \]

with the substitution

\[ y(x) = u(x) \cdot v(x) \quad v(x) = e^{-\frac{1}{2} \int P(x)dx} \]

Use this to convert Bessel’s equation of order \( v \) to normal form

\[ x^2y'' + xy' + \left( x^2 - v^2 \right)y = 0 \]

Another way of calculating the Wronskian of the two solutions \( y_1 \) and \( y_2 \) of \( y'' + P(x)y' + Q(x)y = 0 \) on \((a, b)\) is to use Abel’s Identity:

\[ W(y_1, y_2) = C e^{-\int_{x_0}^x P(t)dt} \quad x_0 \in (a, b) \quad P \text{ and } Q \text{ continuous on } (a, b) \]

This can be easily derived by noticing that if \( \{y_1, y_2\} \) is a fundamental solution set to \( y'' + P(x)y' + Q(x)y = 0 \) on \((a, b)\) then:

\[ y_1'' + Py_1' + Qy_1 = 0 \quad y_2'' + Py_2' + Qy_2 = 0 \]

Multiplying the first equation by \( y_2 \) and the second by \( y_1 \) gives:

\[ y_2y_1'' + Py_2y_1' + Qy_2y_1 = 0 \quad y_1y_2'' + Py_1y_2' + Qy_1y_2 = 0 \]

Subtracting the second equation from the first gives:

\[ y_2y_1'' - y_1y_2'' + P(y_1'y_2 - y_2y_1') = 0 \]

Remembering

\[ W(y_1, y_2) = y_1'y_2 - y_2y_1' \quad \text{and} \quad W'(y_1, y_2) = y_2y_1'' - y_1y_2'' \]

The above equation becomes:

\[ W'(y_1, y_2) + PW(y_1, y_2) = 0 \quad \text{or} \quad \frac{W'(y_1, y_2)}{W(y_1, y_2)} = -P \]

Integrating and solving for \( W(y_1, y_2) \)

\[ \ln |W(y_1, y_2)| = -\int_{x_0}^x P(t)dt + C \]

\[ W(y_1, y_2) = Ce^{-\int_{x_0}^x P(t)dt} \]

This result is known as Abel’s Identity.
254. Use Abel’s Identity to find the Wronskian up to a constant for

\[(1 + x^2)y'' + 2xy' + y = 0 \quad \text{on } (-\infty, \infty)\]

255. Use Abel’s Identity to find the Wronskian up to a constant for

\[(x^2 + 3x + 2)y'' + y' + y = 0 \quad \text{on } (0, \infty)\]

256. Use Abel’s Identity to find the Wronskian up to a constant for

\[
\cos^2(x)y'' + \left( \cos^3(x) + 1 \right)y' + y = 0 \quad \text{on } \left( -\frac{\pi}{4}, \frac{\pi}{4} \right)
\]

257. Use Abel’s Identity to find the Wronskian up to a constant for

\[xy'' + (2x^2 + 1)y' + xy = 0 \quad \text{on } (0, \infty)\]

258. Use Abel’s Identity to find the Wronskian up to a constant for

\[y'' + \frac{6x}{x^4 + 5x^2 + 4}y' + y = 0 \quad \text{on } (-\infty, \infty)\]

259. Use Abel’s Identity to find the Wronskian up to a constant for

\[y'' - \frac{16x}{4x^2 - 1}y' + \frac{16}{4x^2 - 1}y = 0 \quad \text{on } \left( \frac{1}{4}, \infty \right)\]

260. Show \(y_1 = \frac{1}{x-1}\) and \(y_2 = \frac{1}{x+1}\) are solutions to the given differential equation

\[(x^2 - 1)y'' + 4xy' + 2y = 0\]

and calculate the Wronskian of \(y_1\) and \(y_2\) and then confirm your answer with Abel’s Identity.

261. Use Abel’s Identity to find the Wronskian up to a constant for

\[(P(x)y')' + Q(x)y = 0\]

262. If the Wronskian of the solutions to

\[y'' + P(x)y' + Q(x)y = 0\]

is a constant, what does it say about \(P(x)\)
2.2 Reduction of Order

Question: given a second order differential equation of the form:

\[ y'' + P(x)y' + Q(x)y = 0 \]

and one solution to the differential equation can we find a second solution? If \( y_1 \) is a solution to \( y'' + P(x)y' + Q(x)y = 0 \) then we know \( y''_1 + P(x)y'_1 + Q(x)y_1 = 0 \). Let us try to find a second solution of the form \( y_2 = vy_1 \). Differentiating gives:

\begin{align*}
  y'_2 &= vy'_1 + v'y_1 \\
  y''_2 &= vy''_1 + 2v'y'_1 + v''y_1
\end{align*}

Substituting these into the original differential equation gives:

\[ vy''_1 + 2v'y'_1 + v''y_1 + P(x)(vy'_1 + v'y_1) + Q(x)vy_1 = 0 \]

Which reduces to:

\[ v(y''_1 + P(x)y'_1 + Q(x)y_1) + 2v'y'_1 + v''y_1 + P(x)v'y_1 = 0 \]

\[ v''y_1 + 2v'y'_1 + P(x)v'y_1 = 0 \]

\[ v''y_1 + v'(2y'_1 + P(x)y_1) = 0 \]

\[ v''y_1 = -v'(2y'_1 + P(x)y_1) \]

\[ \frac{v''}{v'} = -\frac{2y'_1}{y_1} - P(x) \]

Integrating gives:

\[ \ln |v'| = -2 \ln |y_1| - \int P(x)dx \]

Solving for \( v' \)

\[ v' = e^{\ln |y_1|^{-2} - \int P(x)dx} = \frac{e^{-\int P(x)dx}}{y_1^2} \]

Integrating again give the formula for \( v \)

\[ v = \int \frac{e^{-\int P(x)dx}}{y_1^2} dx \]

An Example: Find a second linearly independent solution:

\[ xy'' - (x + 1)y' + y = 0 \quad y_1 = e^x \]

Writing the equation is standard form gives:

\[ y'' + \left( -1 - \frac{1}{x} \right)y' + \frac{1}{x}y = 0 \]
\[ v = \int e^{\int -\left(1 - \frac{1}{x}\right) \, dx} \, e^{\int \frac{e^{x+\ln(x)}}{e^{2x}} \, dx} = \int x e^{-x} \, dx = -(x + 1)e^{-x} \]

\[ y_2 = y_1 v = e^{x}(-x + 1) e^{-x} = -x - 1 \]

Making the homogenous solution

\[ y_h = C_1 y_1 + C_2 y_2 \]

\[ y_h = C_1 e^{x} + C_2 (x + 1) \]

263. \( y_1 = e^{2x} \) is one solution to \( y'' - 6y' + 8y = 0 \). Use reduction of order to find a second linearly independent solution.

264. \( y_1 = x^{-2} \) is one solution to \( x^2 y'' + 6x y' + 6y = 0 \). Use reduction of order to find a second linearly independent solution.

265. \( y_1 = \frac{1}{x} \) is one solution to \( x y'' + (2x + 2)y' + 2y = 0 \). Use reduction of order to find a second linearly independent solution.

266. \( y_1 = e^{x} \) is one solution to \( x y'' - (x + 1)y' + y = 0 \). Use reduction of order to find a second linearly independent solution.

267. \( y_1 = e^{x} \) is one solution to \( x y'' + (1 - 2x)y' + (x - 1)y = 0 \). Use reduction of order to find a second linearly independent solution.

268. \( y_1 = e^{x} \) is one solution to \( (2x - 1)y'' - (4x^2 + 1)y' + (4x^2 - 2x + 2)y = 0 \). Use reduction of order to find a second linearly independent solution.

269. \( y_1 = e^{x} \) is one solution to \( (\sin(x) - \cos(x))y'' - 2\sin(x)y' + (\sin(x) + \cos(x))y = 0 \). Use reduction of order to find a second linearly independent solution.

270. \( y_1 = x^2 \) is one solution to \( x^2 y'' + (2 - x^2)y' + (2x - 2)y = 0 \). Use reduction of order to find a second linearly independent solution.

271. \( y_1 = \tan(x) \) is one solution to \( y'' - \tan(x)y' - \sec^2(x)y = 0 \). Use reduction of order to find a second linearly independent solution.

272. \( y_1 = x \sin(x) \) is one solution to \( x^2 y'' - 2xy' + (2 + x^2)y = 0 \). Use reduction of order to find a second linearly independent solution.
273. \( y_1 = x + 1 \) is one solution to \((x^2 + 2x - 1)y'' - (2x + 2)y' + 2y = 0\). Use reduction of order to find a second linearly independent solution.

274. \( y_1 = x^2 + 1 \) is one solution to \( y'' - \frac{2x}{x^2 - 1}y' + \frac{2}{x^2 - 1}y = 0 \). Use reduction of order to find a second linearly independent solution.

275. \( y_1 = \sin(x) \) is one solution to \( y'' - 3\cot(x)y' + \frac{3 - 2\sin^2(x)}{\sin^2(x)}y = 0 \). Use reduction of order to find a second linearly independent solution.

276. \( y_1 = \frac{1}{x-2} \) is one solution to \((x^2 - 4)y'' + 4xy' + 2y = 0\). Use reduction of order to find a second linearly independent solution.

277. \( y_1 = x^2 \) is one solution to

\[
y'' - \frac{x^3 - 3x + 1}{x^3 - 3x}y' + \frac{2x^3 - 2x^2 + 2}{x^4 - 3x^3}y = 0
\]

Use reduction of order to find a second linearly independent solution.

278. \( y_1 = x \) is one solution to

\[
y'' - \frac{x}{x - 1}y' + \frac{1}{x - 1}y = 0
\]

Use reduction of order to find a second linearly independent solution.

279. \( y_1 = \frac{\sin(x)}{\sqrt{x}} \) is one solution to the Bessel equation of order \( \frac{1}{2} \):

\[
x^2y'' + xy' + \left( x^2 - \frac{1}{4} \right)y = 0
\]

Use reduction of order to find a second linearly independent solution.

280. \( y_1 = \ln(x) \) is one solution to

\[
x^2 \left( \ln(x) \right)^2 y'' - 2x \ln(x)y' + \left( 2 + \ln(x) \right)y = 0 \quad \text{for} \quad x > 0
\]

Use reduction of order to find a second linearly independent solution.

281. The Hermite equation is an equation of the form:

\[
y'' - 2xy' + \lambda y = 0
\]

Find the homogenous solution for the given values of \( \lambda \) and \( y_1 \)

A) \( \lambda = 4 \) and \( y_1 = 1 - 2x^2 \)
B) \( \lambda = 6 \) and \( y_1 = 3x - 2x^3 \)
The Legendre equation is an equation of the form:

\[(1 - x^2)y'' - 2xy' + \lambda(\lambda + 1)y = 0 \quad x \in (-1, 1)\]

Find the homogeneous solution for the given values of $\lambda$ and $y_1$

A) $\lambda = 1$ and $y_1 = x$
B) $\lambda = 2$ and $y_1 = 3x^2 - 1$
C) $\lambda = 3$ and $y_1 = x^3 - 3x$

The Laguerre equation is an equation of the form:

\[xy'' + (1 - x)y' + \lambda y = 0\]

Find the homogeneous solution for the given values of $\lambda$ and $y_1$

A) $\lambda = 1$ and $y_1 = x - 1$
B) $\lambda = 2$ and $y_1 = x^2 - 4x + 2$

First, use Abel’s Identity to find the Wronskian up to a constant for

\[y'' - \frac{16x}{4x - 1}y' + \frac{16}{4x - 1}y = 0 \quad \text{on} \quad \left(\frac{1}{4}, \infty\right)\]

Second, notice $y_1 = x$ is a solution and apply reduction of order to find $y_2$

The reduction of order algorithm can be applied to third order equations although the formula we derived for $v$ above will not work. $y_1 = e^x$ is a solution to:

\[xy''' - xy'' + y' - y = 0\]

Use $y_2 = vy_1$ to reduce this equation to a second order equation by letting $w = y'$

### 2.3 Equations of the form $y''=f(x,y')$ and $y''=f(y,y')$

For a second order equation of the form:

\[y'' = f(x, y')\]

the substitution $v = y'$ will transform the equation into first order equation.

*An Example:*

Solve:

\[xy'' - y' = 3x^2\]

Using the substitution $v = y'$, making $v' = y''$ our equation becomes:
CHAPTER 2. SECOND ORDER EQUATIONS.

\[ xv' - v = 3x^2 \]  in standard form  \[ v' + \frac{1}{x}v = 3x \]

This equation is now first order linear. Creating an integrating factor

\[ I(x) = e^{\int \frac{1}{x} \, dx} = x^{-1} \]

And now the solution in terms of \( v \)

\[ v = x \left( \int x^3 dx + C \right) \]

\[ v = x(x^3 + C) \]

Converting back to \( y \)

\[ y' = x^4 + Cx \]

Integrating

\[ y = \frac{x^5}{5} + \frac{Cx^2}{2} + K \]

If the differential equation is of the form:

\[ y'' = f(y, y') \]

the substitution \( v = y' \) will transform the equation into a first order equation.

An Example from differential geometry:

The curvature of a circle of radius \( r \) is defined to be \( \frac{1}{r} \) and the curvature of a straight line defined to be zero. For other equations in \( \mathbb{R}^2 \) the curvature is given by the second order equation:

\[ K = \frac{|y''|}{(1 + (y')^2)^{\frac{3}{2}}} \]

By replacing \( K \) with \( \frac{1}{r} \) and solving the second order equation we should obtain the equation of a circle of radius \( r \). For simplicity we will assume \( y'' > 0 \) so we do not have to deal with the pesky absolute values.

\[ \frac{1}{r} = \frac{y''}{(1 + (y')^2)^{\frac{3}{2}}} \]

Substituting \( v = y' \) making \( v' = y'' \) we get:

\[ \frac{1}{r} = \frac{v'}{(1 + (v)^2)^{\frac{3}{2}}} \quad \text{or} \quad \frac{dx}{r} = \frac{dv}{(1 + (v)^2)^{\frac{3}{2}}} \]

This is now a separable and we now need to integrating both sides.
\[ \int \frac{dx}{r} = \int \frac{dv}{(1 + (v^2)^{\frac{3}{2}})} \]

The integral on the left is easy while the integral on the right will require a trig substitution.

\[ \frac{x}{r} + C = \int \frac{dv}{(1 + (v^2)^{\frac{3}{2}})} \quad v = \tan \theta \quad dv = \sec^2 \theta d\theta \]

Under this substitution the integral becomes:

\[ \frac{x}{r} + C = \int \frac{\sec^2 \theta d\theta}{(1 + \tan^2 \theta)^{\frac{3}{2}}} \]

\[ \frac{x}{r} + C = \int \cos \theta d\theta \]

\[ \frac{x}{r} + C = \sin \theta \]

Using the triangle to convert back to \(v\):

\[ \frac{x}{r} + C = \frac{v}{\sqrt{1 + v^2}} \]

Converting back to \(y\) gives another separable equation:

\[ \frac{x + C}{r} = \frac{y'}{\sqrt{1 + (y')^2}} \]

Solving for \(y'\)

\[ (x + C)(\sqrt{1 + y'^2}) = ry' \]

\[ (x + C)^2(1 + (y')^2) = r^2(y')^2 \]

\[ (x + C)^2 + (x + C)^2(y')^2 = r^2(y')^2 \]

\[ (y')^2 = \frac{(x + C)^2}{r^2 - (x + C)^2} \]

\[ y' = \frac{(x + C)}{\sqrt{r^2 - (x + C)^2}} \quad \text{or} \quad \int dy = \int \frac{(x + C)}{\sqrt{r^2 - (x + C)^2}} dx \]

Substituting \(u = r^2 - (x + C)^2\), \(\frac{1}{2} du = (x + C)dx\) for the integral on the right.
\[ y + K = -\frac{1}{2} \int u^{-1} \, du \]

\[ y + K = -\sqrt{r^2 - (x + C)^2} \]

Squaring both sides and rearranging terms gives the equation of a circle of radius \( r \):

\[ (y + K)^2 + (x + C)^2 = r^2 \]

286.
Solve:

\[ (1 + x^2)y'' = 2xy' \]

287.
Solve:

\[ y'' + \frac{1}{x} y' = \frac{4x}{y^2} \quad y'(1) = 2 \]

288.
Solve:

\[ xy'' - y' = 3x^2 \]

289.
Solve:

\[ x^2y'' = 2xy' + (y')^2 \]

290.
Solve:

\[ y'' + x(y')^2 = 0 \]

291.
Solve:

\[ x^2y'' = (y')^2 \]

If the independent variable \( x \) is missing from the differential equation then you will have an equation of the form:

\[ y'' = f(y, y') \]

and the substitution \( v = y' \) will transform the second order equation into a first order equation.

\[ v = \frac{dy}{dx} \quad \frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy} \cdot \frac{dy}{dx} = v \frac{dv}{dy} \]
Using this substitution our equation becomes:

\[ v \frac{dv}{dy} = f(y, v) \]

An Example:
Solve

\[ y'' + k^2 y = 0 \]

Using the substitution above the second order equation becomes:

\[ v \frac{dv}{dy} + k^2 y = 0 \]

This equation is now separable:

\[ vdv = -k^2 ydy \]

Integrating and solving for \( v \):

\[ \frac{v^2}{2} = \frac{-k^2 y^2}{2} + C \]

\[ v = \sqrt{C - k^2 y^2} \quad \text{or} \quad \frac{dy}{dx} = \pm k\sqrt{A^2 - y^2} \]

This equation is now separable:

\[ \int \frac{dy}{\sqrt{A^2 - y^2}} = \int \pm kdx \]

\[ \arcsin \left( \frac{y}{A} \right) = \pm kx + B \]

\[ y = A \sin(\pm kx + B) \]

Solve:

\[ yy'' + (y')^2 = 0 \]

Solve:

\[ yy'' = y^2 y' + (y')^2 \]

Solve:

\[ y'' + 2yy' = y \]
CHAPTER 2. SECOND ORDER EQUATIONS.

295. Solve:

\[ yy'' = (y')^3 \]

296. Solve:

\[ y'' = 12y(y')^2 \]

297. If

\{y_1, (y_1)^2\}

are solutions to

\[ y'' - 3y' + ky = 0 \]

Find \(y_1\)

298. Solve:

\[ W(y, y') = 0 \]

Interprit the results in terms of linear independec or dependence

2.4 Homogenous Linear Equations with Constant Coefficients.

Let us consider the second order equation:

\[ ay'' + by' + cy = 0 \]

and look for a solution of the form \(y = e^{rx}\). Differentiating twice give:

\[ y = e^{rx}, \quad y' = re^{rx}, \quad y'' = r^2e^{rx} \]

Substituting these into the differential equation gives:

\[ ar^2e^{rx} + bre^{rx} + ce^{rx} = 0 \]

Dividing by \(e^{rx}\) gives the characteristic polynomial:

\[ ar^2 + br + c = 0 \]

This quadratic equation can have 3 types of roots: real and distinct, complex or real and repeated.
If the two roots, \( r_1 \) and \( r_2 \) of the characteristic polynomial are real and distinct: \( r_1 \neq r_2 \) then the solution to the differential equation is

\[
y = C_1 e^{r_1 x} + C_2 e^{r_2 x}
\]

If the two roots, \( r_1 \) and \( r_2 \) of the characteristic polynomial are complex \( r_1 = \alpha + \beta i \) and \( r = \alpha - \beta i \) then the solution to the differential equation is

\[
y = C_1 e^{\alpha x} e^{\beta i x} + C_2 e^{\alpha x} e^{-\beta i x}
\]

After applying Euler’s equation: \( e^{i \theta} = \cos(\theta) + i \sin(\theta) \) we get the solution to the differential equation to be:

\[
y = C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x)
\]

If the two roots, \( r_1 \) and \( r_2 \) of the characteristic polynomial are real and repeated: \( r_1 = r_2 = \frac{-b}{2a} \) then one solution to the differential equation is

\[
y_1 = e^{r_1 x}
\]

The second linearly independent solution comes by applying the reduction of order algorithm to the problem. If

\[
y_1 = e^{r_1 x} = e^{\frac{-b}{2a} x} \text{ is a solution to } y'' + \frac{b}{a} y' + \frac{c}{a} y = 0
\]

then the reduction of order algorithm gives:

\[
v = \int \frac{e^{-\int \frac{b}{a} dx}}{(e^{\frac{-b}{2a} x})^2} dx = \int \frac{e^{-\frac{b}{2a} x}}{e^{\frac{-b}{2a} x} dx} = x
\]

Making \( y_2 = xy_1 = xe^{r_1 x} \) and the solution to the differential equation is

\[
y = C_1 e^{r_1 x} + C_2 xe^{r_2 x}
\]

299. Solve

\[
y'' - y' - 2y = 0 \quad y(0) = 2 \quad y'(0) = 1
\]

300. Solve

\[
y'' - 12y' + 36y = 0 \quad y(0) = 1 \quad y'(0) = 1
\]

301. Solve

\[
y'' - 2y' + 10y = 0 \quad y(0) = 2 \quad y'(0) = 1
\]

302. Solve
CHAPTER 2. SECOND ORDER EQUATIONS.

303. Solve
\[ y'' - 8y' + 41y = 0 \quad y(0) = 1 \quad y'(0) = 1 \]

304. Solve
\[ y'' - 16y' + 64y = 0 \quad y(0) = 1 \quad y'(0) = 9 \]

305. Solve
\[ y''' - 6y'' + 11y' - 6y = 0 \quad y(0) = 1 \quad y'(0) = 1 \quad y''(0) = 1 \]

306. Solve
\[ y''' - 7y'' + 15y' - 9y = 0 \]

307. Solve
\[ y''' - 8y'' + 24y' - 32y + 16y = 0 \]

308. Find a third order differential equation with the following solution
\[ y_h = C_1 e^{3x} + C_2 x^3 \sin(2x) + C_3 x^3 \cos(2x) \]

309. Solve the differential equation for different values of \( k \) and sketch the solutions
\[ y'' + ky' + 6y = 0 \quad k \in \{-7, 5, 2, 0\} \]

310. Solve
\[ y'' - 2y' + 3y = 0 \]

311. Consider the differential equation:
\[ ay'' + by' + cy = 0 \quad a, b, c \in \mathbb{R}^+ \]

Show all solutions \( y \) tend to zero as \( x \) tends to infinity. Show this is not true if \( b = 0 \)
2.4. HOMOGENOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS.

312. Consider the differential equation:

\[ ky'' + (k + 1)y' + (k + 2)y = 0 \]

Find the values of \( k \) so that the characteristic polynomial has real and distinct roots, real and repeated roots and complex roots. What values of \( k \) make the solution \( y \) tend to zero as \( x \to \infty \)

313. Consider the differential equation:

\[ y'' + ay' + a^2y = 0 \]

Show the characteristic polynomial has complex roots for all \( a \neq 0 \)

314. Show that if \( y_1 \) and \( y_2 \) are solution to the second order linear equation

\[ ay'' + by' + cy = 0 \quad a, b, c \in \mathbb{R} \]

Then \( y = C_1 y_1 + K y_2 \) is also a solution.

315. Find a third order homogenous differential equation with constant coefficients with the given solution

\[ y = C_1 e^{2x} + C_2 x e^{2x} + C_3 x^2 e^{2x} \]

316. If \( y = xe^x \) is a solution to

\[ y'' + ay' + by = 0 \]

Find \( a \) and \( b \)

317. Solve

\[ \frac{dz}{dt} = e^{(a+bi)t} \]

by separating into real and imaginary parts. Then use the fact that

\[ \int e^{at} \cos(bt)dt = \frac{e^{at}}{a^2 + b^2} \left( a \cos(bt) + b \sin(bt) \right) + C \]

To calculate

\[ \int e^{at} \sin(bt)dt \]

318. For the given differential equation:

\[ y'' + 2y' + Cy = 0 \quad C \in \mathbb{R} \]
A) For what values of $C$ does the characteristic equation have two real distinct roots, real repeated roots and complex roots.

B) For the case where you have two real distinct roots find the values of $C$ that make the solution tend to zero as $x \to \infty$.

Solve the differential equation with the discontinuous coefficient function

$$y'' + sgn(x)y = 0$$

where

$$sgn(x) = \begin{cases} 
-1 & x < 0 \\
1 & 0 < x 
\end{cases}$$

### 2.5 The Method of Undetermined Coefficients

In this section we will study a way to solve the Linear Differential equation:

$$y'' + ay' + by = g(x)$$

for different equations $g(x)$.

For example if you were given the differential equation

$$y'' - 3y' + 5y = 3e^x$$

It would be reasonable to think the solution would involve the exponential function $e^x$ simply because exponential functions differentiate into more exponential functions.

If you were given the differential equation

$$y'' + y + y = \cos(x) + \sin(x)$$

It would be reasonable to think the solution would involve the functions $\sin(x)$ and $\cos(x)$ since $\sin(x)$ and $\cos(x)$ differentiate into each other.

So in the method of undetermined coefficients we guess the form of the solution and try to make the constants work. For

$$y'' - 3y' + 5y = 3e^x$$

we would guess the form of the solution to be $y = Ae^x$ and solve for the value of $A$ that makes it a solution.

For

$$y'' + y' + y = \cos(x) + \sin(x)$$

we would guess the form of the solution to be $y = A\sin(x) + B \cos(x)$ and solve for the values of $A$ and $B$ that makes it a solution.

One problem that can arise using this method is the problem of finding homogenous solution twice. If you were given
it would be reasonable to think the solution would be of the form $y = Ae^{2x}$ but we know the homogenous solution to

$$y'' - 4y' + 4y = 0$$

is $y = C_1e^{2x} + C_2xe^{2x}$. So looking for a particular solution of the form $y = Ae^{2x}$ will again find the homogenous solution, not the particular solution we are interested in. The solution to this problem is to multiply the particular solution by $x^s$ so that it is linearly independent of the terms in the fundamental solution set. The nonnegative integer $s$ should be chosen to be the smallest nonnegative integer so that no term in the particular solution appears in the fundamental solution set. In this problem the fundamental solution set is:

$$\{e^{2x}, xe^{2x}\}$$

So we would choose $s = 2$ and look for a particular solution of the form $y = Ax^2e^{2x}$.

If you choose the wrong form of the particular solution you’re gonna have a bad time.

In general we choose the form of the particular solution based on the following table:

<table>
<thead>
<tr>
<th>$g(x)$</th>
<th>$y_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_n = a_n x^n + ... + a_1 x + a_0$</td>
<td>$x^s P_n = x^s(A_n x^n + ... + A_1 x + A_0)$</td>
</tr>
<tr>
<td>$ae^{\alpha x}$</td>
<td>$Ax^s e^{\alpha x}$</td>
</tr>
<tr>
<td>$a \cos(\beta x) + b \sin(\beta x)$</td>
<td>$x^s (A \cos(\beta x) + B \sin(\beta x))$</td>
</tr>
<tr>
<td>$p_n e^{\alpha x}$</td>
<td>$x^s (P_n e^{\alpha x})$</td>
</tr>
<tr>
<td>$p_n \cos(\beta x) + p_m \sin(\beta x)$</td>
<td>$x^s (P_N \cos(\beta x) + Q_N \sin(\beta x))$ $N = \max(n, m)$</td>
</tr>
<tr>
<td>$ae^{\alpha x} \cos(\beta x) + be^{\alpha x} \sin(\beta x)$</td>
<td>$x^s (A e^{\alpha x} \cos(\beta x) + B e^{\alpha x} \sin(\beta x))$</td>
</tr>
<tr>
<td>$p_n e^{\alpha x} \cos(\beta x) + q_m e^{\alpha x} \sin(\beta x)$</td>
<td>$x^s (P_N e^{\alpha x} \cos(\beta x) + Q_N e^{\alpha x} \cos(\beta x))$ $N = \max(n, m)$</td>
</tr>
</tbody>
</table>

The integer $s$ is chosen to be the smallest nonnegative integer so that no term in the particular solution appears in the homogenous solution.

An Example: Solve:

$$y'' - 4y' + 4y = 6e^{2x}$$

First we need the homogenous solution, so we form the auxiliary equation:

$$r^2 - 4r + 4 = 0 \quad (r - 2)^2 = 0 \quad y_1 = e^{2x} \quad y_2 = xe^{2x}$$

The homogenous solution is:
\[ y_h = C_1 e^{2x} + C_2 xe^{2x} \]

The Fundamental Solution Set is:

\[ F.S.S. = \{ e^{2x}, xe^{2x} \} \]

Due to the right hand side of the differential equation we will choose a particular solution of the form \( y_p = x^s(Ae^{2x}) \) with \( s \) chosen to be the smallest positive integer so that no term in \( y_p \) is in the fundamental solution set. So we choose \( s = 2 \) making \( y_p = Ax^2e^{2x} \)

\[ y_p' = e^{2x}(2Ax + 2Ax^2) \]

Differentiating and simplifying gives:

\[ y_p'' = e^{2x}(4Ax^2 + 8Ax + 2A) \]

Substituting these derivatives into the original differential equation gives:

\[ e^{2x}(4Ax^2 + 8Ax + 2A) - 4e^{2x}(2Ax + 2Ax^2) + 4Ax^2e^{2x} = 6e^{2x} \]

This simplifies nicely to:

\[ 2Ae^{2x} = 6e^{2x} \quad A = 3 \]

So our particular solution is:

\[ y_p = 3x^2e^{2x} \]

The general solution to the differential equation is the sum of the homogenous solution and the particular solution:

\[ y = C_1 e^{2x} + C_2 xe^{2x} + 3x^2e^{2x} \]

320. Solve

\[ y'' + 3y' + 2y = 18 \sin(2x) + 26 \cos(2x) \quad y(0) = 2 \quad y'(0) = 4 \]

321. Solve

\[ y'' + 4y = (5x^2 + 9x + 4)e^x \]

322. Solve:

\[ y'' - 5y' + 6y = e^x(4 \sin(x) - 2 \cos(x)) \quad y(0) = 3 \quad y'(0) = 3 \]

323. Solve
2.5. THE METHOD OF UNDETERMINED COEFFICIENTS

\[ y'' - 4y' + 4y = 6xe^{2x} \quad y(0) = 2 \quad y'(0) = 4 \]

324. Solve

\[ y'' - 4y' + 5y = (2x + 2)e^x \]

325. Solve

\[ y'' + 4y = 14 \cos(x) + 12x \sin(x) \quad y(0) = 2 \quad y'(0) = 4 \]

326. Solve

\[ y'' - 8y' + 7y = -15xe^{2x} - 32e^{2x} \quad y(0) = 2 \quad y'(0) = 4 \]

327. Solve

\[ y'' - 3y' + 2y = e^x \sin(x) \quad y(0) = 2 \quad y'(0) = 4 \]

328. Solve

\[ y'' - 4y' + 4y = 8e^{2x} \]

329. Solve

\[ y'' - 2y' + y = (x + 3)e^x \quad y(0) = 1 \quad y'(0) = 2 \]

330. Solve

\[ y'' + 4y = (5x^2 + 4x + 2)e^x \quad y(0) = 4 \quad y'(0) = 3 \]

331. Solve

\[ y'' - 3y' + 2y = 6x^3 - 19x^2 + 8x - 11 \]

332. Solve

\[ y'' - 3y' + 2y = 2x^2 - 2x - 4 - e^x \quad y(0) = 2 \quad y'(0) = 6 \]

333. Solve
CHAPTER 2. SECOND ORDER EQUATIONS.

Solve

\[ y'' - 4y' + 4y = 16 \sin(x) \cos(x) + 16 \cos^2(x) - 16 \sin^2(x) \]

334.

Solve

\[ y'' + 4y = \cos^3(x) \]

Hint

\[ \cos^3(x) = \frac{3}{4} \cos(x) + \frac{1}{4} \cos(3x) \]

335.

Find the form of the solution to

\[ y'' + 6y' + 10y = x^2 e^{-3x} \sin(x) \]

336.

Find the form of the solution to

\[ y''' - y'' - y' + y = x^3 (e^x + e^{-x}) \]

337.

Find the form of the solution to

\[ y''' - y'' + 4y' - 4y = x^2 (e^x + \sin(2x)) \]

338.

Find a solution of the form \( y = Ax^2 + Bx + C \) to solve the nonlinear differential equation

\[ y'' + (y')^2 + y = 5x^2 + 5x + 3 \]

339.

The form of the particular solution to the third order linear differential equation:

\[ ay''' + by'' + cy' + d = g(x) \]

is

\[ y_p = (Ax^4 + Bx^3 + Cx^2)e^x + Dx e^{2x} \]

what does this tell you about the roots of the characteristic polynomial

\[ ar^3 + br^2 + cr + d = 0 \]

and the function \( g(x) \).
2.6 Variation of Parameters

In the method of variation of parameters we will develop a solution to the second order linear differential equation:

\[ y'' + P(x)y' + Q(x)y = g(x) \]

As you can see the method of variation of parameters is a far more general method of solving differential equation than the method of undetermined coefficients since variation of parameters allows for variable coefficients of \( y' \) and \( y \) whereas the method of undetermined coefficients works only for linear equations with constant coefficients and the function \( g(x) \) does not need to be listed in the table in the section on undetermined coefficients.

Given the fundamental solution set \( \{y_1, y_2\} \) for the homogenous equation:

\[ y'' + P(x)y' + Q(x)y = 0 \]

we will look for a particular solution to the nonhomogeneous equation of the form

\[ y = v_1y_1 + v_2y_2 \]

Differentiating yields:

\[ y' = v_1y_1' + v_1'y_1 + v_2y_2' + v_2'y_2 \]

To avoid second derivatives of the unknown functions \( v_1 \) and \( v_2 \) we impose the condition:

\[ v_1'y_1 + v_2'y_2 = 0 \]

Making

\[ y' = v_1y_1' + v_2y_2' \]

Differentiating again gives:

\[ y'' = v_1y_1'' + v_1'y_1' + v_2y_2'' + v_2'y_2' \]

Substituting \( y, y' \) and \( y'' \) into the original differential equation gives:

\[ v_1y_1'' + v_1'y_1' + v_2y_2'' + v_2'y_2' + P(x)(v_1y_1' + v_2y_2') + Q(x)(v_1y_1 + v_2y_2) = g(x) \]

This equation can be rewritten as:

\[ v_1(y_1'' + P(x)y_1' + Q(x)y_1) + v_2(y_2'' + P(x)y_2' + Q(x)y_2) + y_1'v_1' + y_2'v_2' = g(x) \]

Since \( y_1 \) and \( y_2 \) are solutions of the homogenous equation

\[ y'' + P(x)y' + Q(x)y = 0 \]
the above equation reduces to:

\[ y_1'v_1' + y_2'v_2' = g(x) \]

This equation along with the restriction we made on \( y' \): \( v_1'y_1 + v_2'y_2 = 0 \) gives us a system of two equations with two unknown variables: \( v_1' \) and \( v_2' \) that we will solve using Cramer’s Rule. Writing the system as a matrix equation gives:

\[
\begin{bmatrix}
y_1 \\
y_1'
\end{bmatrix}
\begin{bmatrix}
y_2 \\
y_2'
\end{bmatrix}
=
\begin{bmatrix}
0 \\
g(x)
\end{bmatrix}
\]

Solving for the two unknown variables: \( v_1' \) and \( v_2' \) using Cramer’s Rule gives:

\[ v_1' = -\frac{y_2g(x)}{y_1'y_2' - y_2'y_1'} = -\frac{y_2g(x)}{W(y_1, y_2)} \quad \text{and} \quad v_2' = \frac{y_1g(x)}{y_1'y_2' - y_2'y_1'} = \frac{y_1g(x)}{W(y_1, y_2)} \]

Integrating gives:

\[ v_1 = -\int \frac{y_2g(x)}{y_1'y_2' - y_2'y_1'} \, dx = -\int \frac{y_2g(x)}{W(y_1, y_2)} \, dx \quad \text{and} \quad v_2 = \int \frac{y_1g(x)}{y_1'y_2' - y_2'y_1'} \, dx = \int \frac{y_1g(x)}{W(y_1, y_2)} \, dx \]

An Example: Solve:

\[ y'' + y = \sec(x) \]

First note that this equation cannot be solved by the method of undetermined coefficients due to the fact that the right hand side: \( \sec(x) \) is not in the table in the undetermined coefficients section.

First we need the homogenous solution, so we form the auxiliary equation:

\[ r^2 + 1 = 0 \quad r = \pm i \quad y_1 = \cos(x) \quad y_2 = \sin(x) \]

The homogenous solution is:

\[ y_h = C_1 \cos(x) + C_2 \sin(x) \]

Our fundamental solution set is:

\[ \text{F.S.S.} = \{\cos(x), \sin(x)\} \]

The Wronskian of the fundamental solution set is:

\[ W(\cos(x), \sin(x)) = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = \cos^2(x) + \sin^2(x) = 1 \]

\[ v_1 = -\int \sin(x) \sec(x) \, dx = -\int \tan(x) \, dx = \ln(\cos(x)) \]

\[ v_2 = \int \cos(x) \sec(x) \, dx = x \]
\[ y_p = v_1 y_1 + v_2 y_2 = \ln(\cos(x)) \cos(x) + x \sin(x) \]

So the general solution is:

\[ y = C_1 \cos(x) + C_2 \sin(x) + \ln(\cos(x)) \cos(x) + x \sin(x) \]

340. Solve:

\[ y'' - 3y' + 2y = xe^{2x} \]

341. Solve:

\[ y'' + y = \tan(x) \]

342. Solve:

\[ y'' - 4y' + 5y = xe^{2x} \]

343. Solve:

\[ y'' - 2y' + y = \frac{e^x}{1 + x^2} \]

344. Solve:

\[ y'' + 4y = \sin^3(2x) \]

345. Solve:

\[ y'' - 2y' + y = e^x \arcsin(x) \]

346. Solve:

\[ y'' + y' = \ln(x) \]

347. Solve:

\[ y'' + 4y = \csc(2x) \]
Solve:

\[ y'' + 4y = \ln(x + 1) \]

The fundamental solution set for the differential equation

\[ xy'' - y' - 4x^3y = x^3e^{x^2} \]

is

\[ \{y_1, y_2\} = \{e^{x^2}, e^{-x^2}\} \]

Use variation of parameters to solve the differential equation

Show that \( y_1 = \tan(x) \) is a solution to the homogenous differential equation:

\[ y'' - \tan(x)y' - \sec^2(x)y = 0 \]

Use reduction of order to find the second homogenous solution \( y_2 \) and then use variation of parameters to find the particular and general solution of:

\[ y'' - \tan(x)y' - \sec^2(x)y = \sin(x) \]

Show that \( y_1 = x^3 + x \) is a solution to the homogenous differential equation:

\[ y'' - \frac{4x}{x^2 + 1}y' + \frac{6x^2 - 2}{(x^2 + 1)^2}y = 0 \]

Use reduction of order to find the second homogenous solution \( y_2 \) and then use variation of parameters to find the particular and general solution of:

\[ y'' - \frac{4x}{x^2 + 1}y' + \frac{6x^2 - 2}{(x^2 + 1)^2}y = \frac{2}{1 + x^2} \]

Show that \( y_1 = e^x \) is a solution to the homogenous differential equation:

\[ xy'' - (x + 1)y' + y = 0 \]

Use reduction of order to find the second homogenous solution \( y_2 \) and then use variation of parameters to find the particular and general solution of:

\[ xy'' - (x + 1)y' + y = x^2e^{2x} \]

Show that \( y_1 = x^2 + 1 \) is one solution to
2.6. VARIATION OF PARAMETERS

\[ y'' - \frac{2x}{x^2 - 1} y' + \frac{2}{x^2 - 1} y = 0. \]

Use reduction of order to find the second homogenous solution \( y_2 \) and then use variation of parameters to find the particular and general solution of:

\[ y'' - \frac{2x}{x^2 - 1} y' + \frac{2}{x^2 - 1} y = 2x. \]

354.

Show that \( y_1 = x \) is a solution to the homogenous differential equation:

\[ (x + 1)y'' + xy' - y = 0 \]

Use reduction of order to find the second homogenous solution \( y_2 \) and then use variation of parameters to find the particular and general solution of:

\[ (x + 1)y'' + xy' - y = (x + 1)^2 \]

355.

Show that \( y_1 = x \) is a solution to the homogenous differential equation:

\[ (x^2 - 1)y'' - 2xy' + 2y = 0 \]

Use reduction of order to find the second homogenous solution \( y_2 \) and then use variation of parameters to find the particular and general solution of:

\[ (x^2 - 1)y'' - 2xy' + 2y = x^2 - 1 \]

356.

Show that \( y_1 = 5x - 1 \) is a solution to the homogenous differential equation:

\[ xy'' + (5x - 1)y' - 5y = 0 \]

Use reduction of order to find the second homogenous solution \( y_2 \) and then use variation of parameters to find the particular and general solution of:

\[ xy'' + (5x - 1)y' - 5y = x^2 e^{-5x} \]

357.

Show that \( y_1 = x + 1 \) is a solution to the homogenous differential equation:

\[ (x^2 + 2x)y'' - 2(x + 1)y' + 2y = 0 \]

Use reduction of order to find the second homogenous solution \( y_2 \) and then use variation of parameters to find the particular and general solution of:

\[ (x^2 + 2x)y'' - 2(x + 1)y' + 2y = (x + 2)^2 \]

358.
Show that \( y_1 = \sin(x^2) \) is a solution to the homogenous differential equation:

\[
x y'' - y' + 4x^3 y = 0
\]

Use reduction of order to find the second homogenous solution \( y_2 \) and then use variation of parameters to find the particular and general solution of:

\[
x y'' - y' + 4x^3 y = 2x^3
\]

359.

Show that \( y_1 = \sin(x) \) is a solution to the homogenous differential equation:

\[
\sin^2(x)y'' - 2\sin(x)\cos(x)y' + (1 + \cos^2(x))y = 0
\]

Use reduction of order to find the second homogenous solution \( y_2 \) and then use variation of parameters to find the particular and general solution of:

\[
\sin^2(x)y'' - 2\sin(x)\cos(x)y' + (1 + \cos^2(x))y = \sin^3(x)
\]

360.

Show that \( y_1 = \sin(x) \) is a solution to the homogenous differential equation:

\[
y'' - 3\cot(x)y' + \frac{3 - 2\sin^2(x)}{\sin^2(x)}y = 0
\]

Use reduction of order to find the second homogenous solution \( y_2 \) and then use variation of parameters to find the particular and general solution of:

\[
y'' - 3\cot(x)y' + \frac{3 - 2\sin^2(x)}{\sin^2(x)}y = \sin^3(x)
\]

361.

Solve by first guessing a solution to the homogenous equation then using reduction of order to find the second homogenous solution \( y_2 \) and then use variation of parameters to find the particular and general solution of:

\[
y'' - \frac{2x}{1 + x^2}y' + \frac{2}{1 + x^2}y = 1 + x^2
\]

362.

The Bessel Equation of order one half is:

\[
x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0
\]

and has solutions \( y_1 = \frac{\cos(x)}{\sqrt{x}} \) and \( y_2 = \frac{\sin(x)}{\sqrt{x}} \). Use variation of parameters to solve

\[
x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = x^{\frac{5}{2}}
\]

363.

One solution to the equation:

\[
y'' + P(x)y' + Q(x)y = 0
\]
is \((1 + x)^2\), and the Wronskian of the two solutions is 1. Find the general solution to:

\[ y'' + P(x)y' + Q(x)y = 1 + x \]

### 2.7 Cauchy Euler Equation

The second order Cauchy Euler Equation is:

\[ ax^2 \frac{d^2y}{dx^2} + bx \frac{dy}{dx} + cy = 0 \]

This equation can be transformed into a second order linear differential equation with constant coefficients with use of the substitution:

\[ x = e^t \]

Differentiating with respect to \( t \) gives:

\[ \frac{dx}{dt} = e^t \quad \text{So} \quad \frac{dx}{dt} = x \]

In the Cauchy Euler equation we see the term \( x \frac{du}{dx} \) which we need to substitute for, so I will multiply both sides of \( \frac{dx}{dt} = x \) by \( \frac{dy}{dx} \):

\[ \frac{dx}{dt} \frac{dy}{dx} = x \frac{dy}{dx} \quad \text{This simplifies to} \quad \frac{dy}{dt} = x \frac{dy}{dx} \]

Differentiating \( \frac{dy}{dx} = x \frac{du}{dx} \) with respect to \( x \) gives:

\[ \frac{d}{dx} \frac{dy}{dt} = \frac{dy}{dx} + x \frac{d^2y}{dx^2} \]

In the Cauchy Euler equation we see the term \( x^2 \frac{d^2y}{dx^2} \) which we need to substitute for, so I will multiply the left side of the above equation by \( \frac{dx}{dt} \) and the right hand side by \( x \) (Remember they are equal):

\[ \frac{d}{dx} \frac{dy}{dt} = \left( \frac{dy}{dx} + x \frac{d^2y}{dx^2} \right) x \quad \text{This simplifies to} \quad \frac{d^2y}{dt^2} = x \frac{dy}{dx} + x^2 \frac{d^2y}{dx^2} \]

Replacing \( x \frac{du}{dx} \) with \( \frac{dy}{dt} \) and solving for \( x^2 \frac{d^2y}{dx^2} \) gives:

\[ x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt} \]

So under this substitution the Cauchy Euler equation becomes:

\[ a \left( \frac{d^2y}{dt^2} - \frac{dy}{dt} \right) + b \frac{dy}{dt} + cy = 0 \]

This simplifies to the second order linear equation with constant coefficients:

\[ a \frac{d^2y}{dt^2} + (b - a) \frac{dy}{dt} + cy = 0 \]
Which we can solve by finding the roots of the characteristic polynomial:

\[ ar^2 + (b - a)r + c = 0 \]

Again there are three cases: the roots are real and distinct, the roots are real and repeated or the roots are complex.

Case 1: we have two real and distinct roots \( r_1 \) and \( r_2 \). Then the solutions to the differential equation are:

\[ y_1 = e^{r_1 t} \quad \text{and} \quad y_2 = e^{r_2 t} \]

Since \( x = e^t \), \( t = \ln(x) \) Making the solution:

\[ y_1 = e^{r_1 \ln(x)} = e^{\ln(x^{r_1})} \quad \text{and} \quad y_2 = e^{r_2 \ln(x)} = e^{\ln(x^{r_2})} \]

So

\[ y_1 = x^{r_1} \quad \text{and} \quad y_2 = x^{r_2} \]

Case 2: The roots are real and repeated \( r_1 = r_2 \). Then the solutions to the differential equation are:

\[ y_1 = e^{r_1 t} \quad \text{and} \quad y_2 = te^{r_1 t} \]

Since \( x = e^t \), \( t = \ln(x) \) Making the solution:

\[ y_1 = e^{r_1 \ln(x)} = e^{\ln(x^{r_1})} \quad \text{and} \quad y_2 = \ln(x)e^{r_1 \ln(x)} = \ln(x)e^{\ln(x^{r_1})} \]

So

\[ y_1 = x^{r_1} \quad \text{and} \quad y_2 = \ln(x)x^{r_1} \]

Case 3: The roots are complex \( r_1 = \alpha + \beta i \) and \( r_2 = \alpha - \beta i \). Then the solutions to the differential equation are:

\[ y_1 = e^{\alpha t} \cos(\beta t) \quad \text{and} \quad y_2 = e^{\alpha t} \sin(\beta t) \]

Since \( x = e^t \), \( t = \ln(x) \) Making the solution:

\[ y_1 = x^\alpha \cos(\beta \ln(x)) \quad \text{and} \quad y_2 = x^\alpha \sin(\beta \ln(x)) \]

An Example: Solve:

\[ x^2 y'' - 5xy' + 13y = 0 \]

Forming the characteristic polynomial

\[ r^2 + (-5 - 1)r + 13 = 0 \quad r^2 - 6r + 13 = 0 \quad (r - 3)^2 = -4 \quad r = 3 \pm 2i \]

So we have complex roots so the solution is:

\[ y_h = C_1 e^{3t} \cos(2t) + C_2 e^{3t} \sin(2t) \]

Converting back to \( x \) gives:
$$y_h = C_1x^3\cos(2\ln(x)) + C_2x^3\sin(2\ln(x))$$

An Example: Solve:

$$\sin^2(x)\frac{d^2y}{dx^2} + \tan(x)\frac{dy}{dx} - k^2\cos^2(x)y = 0$$

Although this is not the Cauchy-Euler equation we can transform it into one using the following substitution:

$$u = \sin(x)$$

Using the product rule gives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\cos(x)\frac{dy}{du}\right) = -\sin(x)\frac{dy}{du} + \cos(x)\frac{d}{dx}\left(\frac{dy}{du}\right)$$

$$\frac{d^2y}{dx^2} = -\sin(x)\frac{dy}{du} + \cos(x)\frac{d^2y}{du^2}\frac{du}{dx}$$

$$\frac{d^2y}{dx^2} = -\sin(x)\frac{dy}{du} + \cos^2(x)\frac{d^2y}{du^2}$$

Substituting these expressions into the differential equation gives

$$\sin^2(x)\left(-\sin(x)\frac{dy}{du} + \cos^2(x)\frac{d^2y}{du^2}\right) + \tan(x)\cos(x)\frac{dy}{du} - k^2\cos^2(x)y = 0$$

Rearranging the terms gives

$$\sin^2\cos^2(x)\frac{d^2y}{du^2} + \left(\sin(x) - \sin^3(x)\right)\frac{dy}{du} - k^2\cos^2(x)y = 0$$

$$\sin^2\cos^2(x)\frac{d^2y}{du^2} + \sin(x)\left(1 - \sin^2(x)\right)\frac{dy}{du} - k^2\cos^2(x)y = 0$$

After a trig identity we see a $\cos^2(x)$ in each term that we can eliminate yielding

$$\sin^2\frac{d^2y}{du^2} + \sin(x)\frac{dy}{du} - k^2y = 0$$

Since $u = \sin(x)$ we get the Cauchy-Euler equation

$$\frac{d^2y}{du^2} + \frac{dy}{du} - k^2y = 0$$

Now the Cauchy-Euler substitution

$$u = e^t \quad \frac{dy}{du} = \frac{dy}{dt} \quad u\frac{d^2y}{du^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$
Our differential equation now becomes
\[ \frac{d^2y}{dt^2} - k^2y = 0 \]
which has the characteristic equation
\[ r^2 - k^2 = 0 \quad r = \pm k \]
So the solution is
\[ y = C_1 e^{kt} + C_2 e^{-kt} \]
Converting back to the variable \( u \)
\[ y = C_1 u^k + C_2 u^{-k} \]
Converting back to the variable \( x \) gives the final solution
\[ y = C_1 \sin^k(x) + C_2 \sin^{-k}(x) \]

364. Solve:
\[ x^2 \frac{d^2y}{dx^2} + 7x \frac{dy}{dx} + 8y = 0 \]

365. Solve:
\[ x^2 \frac{d^2y}{dx^2} + 9x \frac{dy}{dx} + 12y = 0 \]

366. Solve:
\[ x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 20y = 0 \]

367. Solve:
\[ x^2 \frac{d^2y}{dx^2} - 11x \frac{dy}{dx} + 36y = 0 \]

368. Solve:
\[ x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^3 \ln(x) \]

369. Solve:
2.7. **CAUCHY EULER EQUATION**

\[ x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} - 8y = (\ln(x))^3 - \ln(x) \]

370. Solve:

\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = x \]

371. Solve:

\[ x^2 \frac{d^2 y}{dx^2} - 5x \frac{dy}{dx} + 8y = x^3 \arctan(x) \]

372. Solve:

\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + 4y = \frac{1}{x} \]

373. Solve:

\[ x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} + y = 8x \]

374. Solve:

\[ x^2 \frac{d^2 y}{dx^2} - 7x \frac{dy}{dx} + 16y = x^3 \]

375. Solve:

\[ x^2 \frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + 6y = \sqrt{1 - x^2} \]

376. Solve:

\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + 4y = 8 \]

377. Solve:

\[ x^2 \frac{d^2 y}{dx^2} + x^2 \frac{dy}{dx} + xy = 0 \]

378. Find a Cauchy Euler equation with the following solution

\[ y = \frac{C_1 x^2 + C_2}{x} \]

379.
Find a Cauchy Euler equation with the following solution

\[ y = C_1 x^2 \cos(4 \ln(x)) + C_2 x^2 \sin(4 \ln(x)) \]

380.
Solve:

\[ xy'' + y' = 0 \]

381.
Solve:

\[ 2(x - 4)^2 \frac{d^2 y}{dx^2} + 5(x - 4) \frac{dy}{dx} - 2y = 0 \]

382.
Solve:

\[ x^3 \frac{d^3 y}{dx^3} - 4x^2 \frac{d^2 y}{dx^2} + 8x \frac{dy}{dx} - 8y = 4 \ln(x) \]

383.
Find the values of \( \alpha \) that make the solution: \( y \), to the given equation tend to zero as \( x \to \infty \)

\[ x^2 \frac{d^2 y}{dx^2} + \alpha x \frac{dy}{dx} + \frac{1}{4} y = 0 \]

384.
Find the value of \( \alpha \) that makes \( y \to 0 \) as \( x \to 0 \)

\[ x^2 \frac{d^2 y}{dx^2} - 6y = 0 \]

\[ y(1) = 1 \quad y'(1) = \alpha \]

385.
The nonlinear equation

\[ y'' + y^2 + P(x)y + Q(x) = 0 \]

is an example of a Riccati Equation. Show the substitution \( y = \frac{z'}{z} \) transform the equation to

\[ z'' + P(x)z' + Q(x)z = 0 \]

386.
Use the results of the previous problem to solve

\[ y' + y^2 - \frac{4}{x} y + \frac{6}{x^2} = 0 \]

387.
Use the results of the previous problem to solve
2.8. EVERYONE LOVES A SLINKY: SPRINGS

\[ y' + y^2 - \frac{4}{x}y + \frac{12}{x^2} = 0 \]

388.

Use the substitution \( x = \sqrt{t} \) to transform the differential equation to an equation with constant coefficients and then solve.

\[ y'' - \frac{1}{x}y' + 4x^2y = 0 \]

389.

Show that if \( y(x) \) is a solution to the Cauchy Euler Equation for \( x > 0 \) then \( y(-x) \) is a solution for \( x < 0 \)

2.8 Everyone Loves a Slinky: Springs

The goal of this section is to develop a differential equation that governs the motion of a mass connected to an ideal spring. We will first study the theoretical case of a spring with no damping: (internal resistance of the spring, air friction etc.). We will also study springs with damping and then with a forcing function attached to the mass.

Newton’s Second law states that force equals mass times acceleration: \( F = ma \). So if \( y(t) \) represents the position of a moving mass on a spring then its acceleration is \( a = \frac{d^2y}{dt^2} \).

Now consider a mass spring system with a mass \( m \) attached and stretched so that the mass is still. This unmoving system is said to be in equilibrium. We measure the distance, \( y \), to be the displacement of the mass from equilibrium. When the mass is displaced from equilibrium, the spring is stretched or compressed and it exerts a force in the opposite direction of the displacement. The force exerted by the spring is given by Hooke’s Law:

\[ F_{spring} = -ky \quad k > 0 \]

where \( k \) is a constant dependent on the stiffness of the spring and \( y \) is the displacement of the mass from equilibrium.

All mass spring systems experience some form of internal resistance known as damping which is proportional to the velocity of the mass. Since the mass has a velocity \( v = \frac{dy}{dt} \) the damping force is given by:

\[ F_{damping} = -b\frac{dy}{dt} \quad b > 0 \]

where \( b \) is the damping coefficient.

All other forces on the system are external making the differential equation governing the mass spring system:

\[ m\frac{d^2y}{dx^2} = -b\frac{dy}{dt} - ky + F_{external}(t) \]

Or

\[ m\frac{d^2y}{dx^2} + b\frac{dy}{dt} + ky = F_{external}(t) \]

In the absence of damping and an external force: \( b = 0, F_{external}(t) = 0 \) the differential equation becomes:

\[ m\frac{d^2y}{dx^2} + ky = 0 \]
Which has an auxiliary \( mr^2 + k = 0 \) which has purely imaginary roots \( r = \pm \omega i \) making the solution:

\[
y = C_1 \cos(\omega t) + C_2 \sin(\omega t) \quad \omega = \sqrt{\frac{k}{m}}
\]

This solution can be written in the form:

\[
y = A \sin(\omega t + \phi)
\]

by first applying a trig identity to \( \sin(\omega t + \phi) \).

\[
y = A \sin(\omega t + \phi) = A(\sin(\omega t) \cos(\phi) + \cos(\omega t) \sin(\phi))
\]

These two solutions are equal if

\[
A \sin(\phi) = C_1 \quad \text{and} \quad A \cos(\phi) = C_2
\]

We see the amplitude of the solution: \( A \) is given by:

\[
A = \sqrt{C_1^2 + C_2^2} \quad \text{and} \quad \tan(\phi) = \frac{C_1}{C_2}
\]

We see the solution is a sinusoid with angular frequency \( \omega = \sqrt{\frac{k}{m}} \) and period \( T = \frac{2\pi}{\omega} \).

All springs experience some form of damping. To explore the nature of this damping let us consider the equation:

\[
my'' + by' + ky = 0
\]

The auxiliary equation is:

\[
mr^2 + br + k = 0
\]

with roots:

\[
r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} = \frac{-b}{2m} \pm \frac{1}{2m} \sqrt{b^2 - 4mk}
\]

The nature of the solution depends on the discriminate \( b^2 - 4mk \).

If \( b^2 - 4mk < 0 \) the roots will be complex and we say the spring system has Underdamped Motion.

Letting \( \alpha \) be the real part and \( \beta \) the imaginary part of the roots we have:

\[
\alpha = \frac{-b}{2m} \quad \text{and} \quad \beta = \frac{1}{2m} \sqrt{4mk - b^2}
\]

The solution is:

\[
y = e^{\alpha t}(C_1 \cos(\beta t) + C_2 \sin(\beta t))
\]

We can express this solution in an alternate form as we did earlier:

\[
y = Ae^{\alpha t} \sin(\beta t + \phi) \quad A = \sqrt{C_1^2 + C_2^2} \quad \text{and} \quad \tan(\phi) = \frac{C_1}{C_2}
\]

So our solution is the product of a sinusoid: \( \sin(\beta t + \phi) \) and an exponential damping factor: \( Ae^{\alpha t} \). As \( t \to \infty \) \( Ae^{\alpha t} \to 0 \) and our solution also tends to zero. Further as \( b \to 0 \) \( \alpha = \frac{-b}{2m} \to 0 \) and the solution tends to the sinusoid:
\[ y = A \sin(\beta t + \phi) \]

Going back to the discriminate: \( b^2 - 4mk \). If \( b^2 - 4mk > 0 \) the roots will real and distinct and we say the spring system has **Overdamped Motion**. The roots to the auxiliary equation are:

\[ r_1 = \frac{-b}{2m} + \frac{1}{2m} \sqrt{b^2 - 4mk} \quad r_2 = \frac{-b}{2m} - \frac{1}{2m} \sqrt{b^2 - 4mk} \]

And the solution is:

\[ y = C_1 e^{r_1 t} + C_2 e^{r_2 t} \]

It is clear that \( r_2 \) is negative; \( r_1 \) is negative as well since \( b^2 > b^2 - 4mk \) making \( b > \sqrt{b^2 - 4mk} \). Subtracting \( b \) from this inequality and dividing by \( 2m \) gives \( r_1 = \frac{-b}{2m} + \frac{1}{2m} \sqrt{b^2 - 4mk} < 0 \).

Since both \( r_1 \) and \( r_2 \) are negative the solution \( y \) will approach zero as \( t \to \infty \).

Going back to the discriminate: \( b^2 - 4mk \). If \( b^2 - 4mk = 0 \) the roots will real and repeated and we say the spring system has **Critically Damped Motion**. The roots to the auxiliary equation are:

\[ r_1 = r_2 = \frac{-b}{2m} \]

And the solution is:

\[ y = e^{-\frac{b}{2m}} (C_1 + C_2 t) \]

As \( t \to \infty \) our solution will tend to zero.

**An Example:**

A 1kg mass hangs from a spring stretching it .392m from equilibrium. The mass is then pulled down .5m and released. Find the equation of the motion of the mass if:

1) damping constant \( b = 0 \)

2) damping constant \( b = 8 \)

3) damping constant \( b = 10 \)

4) damping constant \( b = 26 \)

**Solution:** We already know mass \( m \) and damping constant \( b \) so all we need is the spring constant \( k \). Using Hook’s Law:

\[ 9.8 = k(0.392) \quad k = 25 \]

The differential equation governing the system is:

\[ y'' + by' + 25y = 0 \quad y(0) = -0.392 \quad y'(0) = 0 \]

1) If there is no damping then \( b = 0 \) and our equation becomes:

\[ y'' + 25y = 0 \]
The auxiliary equation and roots are:

\[ r^2 + 25 = 0 \quad r = \pm 5i \]

The solution is:

\[ y = C_1 \cos(5t) + C_2 \sin(5t) \]

After the laborious task of applying the initial conditions we get:

\[ y = -.392 \cos(5t) \]

We see the Amplitude is \( A = .392 \) and \( \phi = \frac{\pi}{2} \). The solution can be written in the form:

\[ y = -.392 \sin \left( 5t + \frac{\pi}{2} \right) \]

2) The damping constant is \( b = 8 \). Our equation becomes:

\[ y'' + 8y' + 25y = 0 \]

The auxiliary equation and roots are:

\[ r^2 + 8r + 25 = 0 \quad r = -4 \pm 3i \]

The solution is:

\[ y = C_1 e^{-4t} \cos(3t) + C_2 e^{-4t} \sin(3t) \]

After the laborious task of applying the initial conditions we get:

\[ y = -.392e^{-4t} \cos(3t) - .52267e^{-4t} \sin(3t) \]

We see the Amplitude is \( A = .653336 \) and \( \phi = .643498 \). Since the initial displacement is down (against the force of the spring) we will use \( A = -.653336 \). The solution can be written in the form:

\[ y = -.653336e^{-4t} \sin(3t + .643498) \]

3) The damping constant is \( b = 10 \). Our equation becomes:

\[ y'' + 10y' + 25y = 0 \]

The auxiliary equation and roots are:

\[ r^2 + 10r + 25 = 0 \quad r = -5 \text{ is repeated root} \]

The solution is:

\[ y = C_1 e^{-5t} + C_2 te^{-5t} \]

After the laborious task of applying the initial conditions we get:
4) The damping constant is \( b = 26 \). Our equation becomes:

\[
y'' + 26y' + 25y = 0
\]

The auxiliary equation and roots are:

\[
r^2 + 26r + 25 = 0 \quad r = -1, -25
\]

The solution is:

\[
y = C_1 e^{-t} + C_2 e^{-26t}
\]

After the laborious task of applying the initial conditions we get:

\[
y = .40768e^{-t} - .01568e^{-26t}
\]

390.

A 3kg mass is attached to a spring with stiffness \( k = 48\,\text{N/m} \). The mass is displaced 1/2m to the left of equilibrium point and given a velocity of 2m/s to the right. The damping constant is 0. Find the equation of motion of the mass along with the amplitude, period, and frequency.

391.

A 1/8kg mass is attached to a spring with stiffness \( k = 16\,\text{N/m} \). The mass is displaced 3/4m to the left of equilibrium point and given a velocity of 2m/s to the left. The damping constant is 2Ns/m. Find the equation of motion of the mass.

392.

Consider the following differential equation

\[
y'' + ty' + y = 0
\]

Although we cannot solve this differential equation we can determine the nature of the solution for large \( t \) by thinking of the equation in terms of a mass spring system.

Find

\[
\lim_{t \to \infty} y(t)
\]

Now let us consider the mass spring system with a forcing function applied to the system. The differential equation governing this system is:

\[
my'' + by' + ky = F_0 \cos(\omega t) \quad F_0 > 0 \quad \omega > 0
\]

Let us first explore the underdamped case \((0 < b^2 < 4mk)\). From previous discussion we know the homogenous solution is:

\[
y_h = Ae^{-\frac{b}{2m}t} \sin\left(\frac{\sqrt{4mk - b^2}}{2m}t + \phi\right)
\]
With

\[ A = \sqrt{C_1^2 + C_2^2} \quad \tan(\phi) = \frac{C_1}{C_2} \]

To find the particular solution we apply the method of undetermined coefficients. We choose the form of the particular solution to be:

\[ y_p = A_1 \cos(\omega t) + A_2 \sin(\omega t) \]

Making:

\[ y'_p = -A_1 \omega \sin(\omega t) + A_2 \omega \cos(\omega t) \quad y''_p = -A_1 \omega^2 \cos(\omega t) - A_2 \omega^2 \sin(\omega t) \]

Substituting this into the differential equation and simplifying gives:

\[ \left( (k - m\omega^2)A_1 + A_2 b\omega \right) \cos(\omega t) + \left( (k - m\omega^2)A_2 + A_1 b\omega \right) \sin(\omega t) = F_0 \cos(\omega t) \]

Equating corresponding coefficients gives:

\[ (k - m\omega^2)A_1 + A_2 b\omega = F_0 \quad (k - m\omega^2)A_2 + A_1 b\omega = 0 \]

Solving this system of equations gives:

\[ A_1 = \frac{F_0(k - m\omega^2)}{(k - m\omega^2)^2 + b^2\omega^2} \quad A_2 = \frac{F_0 b\omega}{(k - m\omega^2)^2 + b^2\omega^2} \]

Making the solution:

\[ y_p = \frac{F_0}{(k - m\omega^2)^2 + b^2\omega^2} \left( (k - m\omega^2) \cos(\omega t) + b\omega \sin(\omega t) \right) \]

Let

\[ \tan(\theta) = \frac{A_1}{A_2} \]

Drawing a triangle for \( \tan(\theta) = \frac{k-m\omega^2}{b\omega} \)

![Diagram](image)

From the triangle we see:

\[ \sin(\theta) = \frac{k - m\omega^2}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}} \quad \cos(\theta) = \frac{b\omega}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}} \]

Making:

\[ k - m\omega^2 = \sqrt{(k - m\omega^2)^2 + b^2\omega^2 \sin(\theta)} \quad b\omega = \sqrt{(k - m\omega^2)^2 + b^2\omega^2 \cos(\theta)} \]

Now our solution is:
2.8. EVERYONE LOVES A SLINKY: SPRINGS

\[ y_p = \frac{F_0}{(k - m\omega^2)^2 + b^2\omega^2} \left( \sqrt{(k - m\omega^2)^2 + b^2\omega^2} \sin(\theta) \cos(\omega t) + \sqrt{(k - m\omega^2)^2 + b^2\omega^2} \cos(\theta) \sin(\omega t) \right) \]

\[ y_p = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}} \left( \sin(\theta) \cos(\omega t) + \cos(\theta) \sin(\omega t) \right) \]

After a trig identity we get:

\[ y_p = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}} \sin(\omega t + \theta) \]

So the general solution is:

\[ y = y_h + y_p = Ae^{-\frac{b}{2m}t} \sin \left( \frac{\sqrt{4mk - b^2}}{2m} t + \phi \right) + \frac{F_0}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}} \sin(\omega t + \theta) \]

The first term in the solution \( y_h \) tends to zero as \( t \) tends to infinity. So we refer to this term as the transient solution. As \( t \) gets large and \( y_h \) approaches zero the solution approaches the particular solution \( y_p \). Hence we call this term the steady state solution.

The factor:

\[ \frac{1}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}} \]

in the particular solution represents the ratio of the magnitude of the forcing function \( F_0 \) to the magnitude of the sinusoidal response to the input force so we call it: the frequency gain and has units length/force.

An Example:

An 8-kg mass is attached to a spring hanging from the ceiling causing the spring to stretch 1.96m upon coming to rest at equilibrium. At \( t = 0 \) the forcing function \( F(t) = \cos(2t) \) is applied to the system. The damping constant for the system is 3 N-sec/m. Find the steady state solution and the frequency gain.

Solution:

At \( t = 0 \) the system is equilibrium so it has an initial position \( y(0) = 0 \) and initial velocity \( y'(0) = 0 \)

First we need the spring constant \( k \). Since the 8-kg mass stretched the spring 1.96m Hooks Law gives:

\[ 8(9.8) = k(1.96) \quad k = 40 \]

The differential equation governing the system is:

\[ 8y'' + 2y' + 40y = \cos(2t) \]

We could use the method of undetermined coefficients to find the steady state solution but we have already derived equations for it:

\[ y_p = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}} \sin(\omega t + \theta) \quad \tan(\theta) = \frac{k - m\omega^2}{b\omega} \]

Plugging in the values for \( m, b, k, F_0, \theta \) and \( \omega \) gives:

\[ \tan(\theta) = \frac{1}{2} \quad \theta = .463648 \]

And the steady state solution is:
(2t + .463648)

The frequency gain is:

\[ \frac{1}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}} = \frac{1}{80} \]

393.

An 8-kg mass is attached to a spring hanging from the ceiling causing the spring to stretch 7.84m upon coming to rest at equilibrium. At \( t = 0 \) the forcing function \( F(t) = 2\cos(2t) \) is applied to the system. The damping constant for the system is 1 N-sec/m. Find the steady state solution and the frequency gain.

394.

An 2-kg mass is attached to a spring hanging from the ceiling causing the spring to stretch .2m upon coming to rest at equilibrium. At \( t = 0 \) the mass is displaced .005m below equilibrium and released. At \( t = 0 \) the forcing function \( F(t) = .3\cos(t) \) is applied to the system. The damping constant for the system is 5 N-sec/m. Find the steady state solution and the frequency gain.

395.

Although we cannot solve the following differential equation

\[ y'' + e^t y' + y = 0 \]

we can determine the limiting behavior of the solution by thinking about the differential equation in terms of a spring equation.

Find

\[ \lim_{t \to \infty} y(t) \]

For a given mass spring system with known mass: \( m \), damping constant: \( b \) and spring constant \( k \) with a forcing function \( F_0\cos(\omega t) \) we know the frequency gain: \( M(\omega) \), will be a be a function of the variable \( \omega \) and is given by:

\[ M(\omega) = \frac{1}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}} \]

Often times we want to know the value of \( \omega \) that maximizes the frequency gain. Differentiating gives:

\[ M'(\omega) = \frac{-1}{2} \left( (k - m\omega^2)^2 + b^2\omega^2 \right)^{-3/2} \left( 2(k - m\omega^2)(-2m\omega) + 2b^2\omega \right) \]

Simplifying:

\[ M'(\omega) = \omega \left( \frac{-2m^2\omega^2 - b^2 + 2km}{(k - m\omega^2)^2 + b^2\omega^2} \right) \]

So \( M \) has critical numbers:

\[ \omega = 0 \quad \text{in this case the forcing function is constant} \]

The critical number we care about is:
\[ \omega = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}} \]

And the maximum frequency gain occurs when:

\[ \omega = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}} \]

The maximum Amplitude of the steady state response is:

\[ F_0M(\omega) = F_0 \frac{1}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}} \]

396.

Let the following differential equation govern the motion of a mass on a spring.

\[ \frac{1}{2} y'' + by' + 10y = 3 \cos(2t) \]

Find the value of \( b \) that maximizes the amplitude of the steady state response.

### 2.9 Circuits

We now turn our study to the circuit consisting of a resistor whose letter representation is \( R \) and is measured in Ohms, a capacitor whose letter representation is \( C \) and the inductor whose letter representation is \( L \) connected to a voltage source whose letter representation is \( E \).

This circuit will be governed by Kirchhoff’s loop rules. Here they are:

1. The sum of the currents flowing into any junction point are zero.

2. The sum of the voltage around any closed loop is zero.

From physics we can find the voltage drop by the resistor, capacitor and the inductor. Here they are

1. The voltage drop across a resistor is given by

   \[ E_R = IR \quad \text{where} \ I \ \text{is the current passing through the resistor} \]

2. The voltage drop across a capacitor is
\[ E_C = \frac{1}{C} Q \quad \text{where } Q \text{ is the charge on the capacitor} \]

3 The voltage drop across an inductor is

\[ E_L = L \frac{dI}{dt} \]

If a voltage source is connected to the circuit an adds voltage at a level of \( E(t) \) then Kirchhoff’s voltage law gives:

\[ E_L + E_R + E_C = E(t) \]

Or

\[ L \frac{dI}{dt} + RI + \frac{1}{C} Q = E(t) \]

Since the change in charge is the current we have

\[ \frac{dQ}{dt} = I \]

Making

\[ \frac{dI}{dt} = \frac{d^2 Q}{dt^2} \]

Now our differential equation becomes

\[ L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C} Q = E(t) \]

Sometimes we want to determine the current \( I(t) \) in the circuit so we differentiate the above equation to get:

\[ L \frac{d^2 I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I = E'(t) \]

397.

A RLC circuit has a voltage source given by \( E(t) = 40 \cos(2t) \) \( V \) a resistor of 2 ohms, a inductor of .25 henrys and a capacitor of \( \frac{1}{13} \) farads. If the initial current is zero and the initial charge on the capacitor is 3.5 coulombs, determine the charge on the capacitor as a function of \( t \).

398.

A RLC circuit with no voltage source has a resistor of 20 ohms, a inductor of .1 henrys and a capacitor of \( \frac{1}{25} \) farads. If the initial current is zero and the initial charge on the capacitor is 10 coulombs, determine the charge on the capacitor as a function of \( t \).

399.

A RLC circuit has a voltage source given by \( E(t) = 40V \) a resistor of 10 ohms, a inductor of .2 henrys and a capacitor of \( \frac{1}{13} \) farads. If the initial current is zero and the initial charge on the capacitor is 0, determine the current as a function of \( t \).
Chapter 3

Series Solution

In calculus we learned that all continuously differentiable function can be represented by a Taylor series. The Taylor series for a function centered at \( x = c \) is:

\[
f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)(x - c)^2}{2!} + \frac{f'''(c)(x - c)^3}{3!} + \frac{f^{(4)}(c)(x - c)^4}{4!} + ... \]

If the series is centered at zero then we call it a Mclauren series.

Some common power series are:

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!}
\]

\[
\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}
\]

\[
\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}
\]

\[
\frac{1}{1 - x} = 1 + x + x^2 + x^3 + ... = \sum_{n=0}^{\infty} x^n
\]

\[
arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n + 1}
\]

3.1 Series Solutions Around Ordinary Points

In this chapter we will not be looking for some equation \( f(x) \) that is the solution to a differential equation, instead we will be looking for its Power Series (normally a Taylor Series). For the case of the Taylor Series we will look for a solution to the differential equation of the form:

\[
y = \sum_{n=0}^{\infty} a_n x^n
\]
with the sequence $a_n$ to be determined by substituting $y$, $y'$ and $y''$ into the differential equation and developing a recurrence relation for $a_n$ and solving the recurrence relation for a formula for $a_n$.

**An Example:**

Find at least the first seven terms in the power series that is a solution to the differential equation

$$y'' + 3xy' + 2y = 0$$

We will look for a solution of the form:

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \text{making} \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

Substituting $y$, $y'$ and $y''$ into the differential equation produces:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 3 \sum_{n=1}^{\infty} n a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

I will now shift the first series in the above expression so that it too has an $x^n$ factor:

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + 3 \sum_{n=1}^{\infty} n a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

Now I will add the zero term in the first and third series so that all indexes will be $n = 1$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2} x^n + 3 \sum_{n=1}^{\infty} n a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

Now that all three series have an $x^n$ factor and all 3 indexes are the same: $n = 1$, we can combine the three series into one series:

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} \left((n+2)(n+1)a_{n+2} + (3n+2)a_n\right) x^n = 0$$

Setting the constant term on the left: $2a_2 + 2a_0$ equal to the constant term on the right: 0 and setting the coefficient on $x^n$ on the left: $(n+2)(n+1)a_{n+2} + (3n+2)a_n$ equal to the coefficient of $x^n$ on the right: 0 gives:

$$2a_2 + 2a_0 = 0 \quad (n+2)(n+1)a_{n+2} + (3n+2)a_n = 0$$

$$a_2 = -a_0 \quad \text{and our recurrence relation is:} \quad a_{n+2} = \frac{-(3n+2)a_n}{(n+2)(n+1)}$$

Substituting $n = 1, 2, ... 5$ in to the recurrence relation gives:

$$a_3 = -\frac{5}{6} a_1 \quad a_4 = \frac{2}{3} a_0 \quad a_5 = \frac{11}{24} a_1 \quad a_6 = -\frac{14}{45} a_0$$

The solution is

$$y = a_0 + a_1 x - a_0 x^2 - \frac{5}{6} a_1 x^3 + \frac{2}{3} a_0 x^4 + \frac{11}{24} a_1 x^5 - \frac{14}{45} a_0 x^6$$

Separating this solution into two linearly independent solutions:
3.1. SERIES SOLUTIONS AROUND ORDINARY POINTS

\[ y = a_0 \left( 1 - x^2 + \frac{2}{3} x^4 - \frac{14}{45} x^6 \right) + a_1 \left( x - \frac{5}{6} x^3 + \frac{11}{24} x^5 \right) \]

Since a pattern in the terms cannot be found we shall leave the solution as a sixth degree polynomial.

Sometimes a pattern can be found and the solution can be written much more concisely. Consider the following example:

**Example:** Solve using power series about \( x = 0 \):

\[(x + 1)y'' - 2xy' - 4y = 0\]

First let's note that \( y = e^{2x} \) is a solution. We will try to find this solution using power series. The series expansion for \( e^{2x} \) that we are trying to obtain is:

\[ e^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = 1 + 2x + 2x^2 + \frac{4}{3} x^3 + \frac{2}{3} x^4 + \frac{4}{15} x^5 \ldots \]

We will look for a solution of the form:

\[ y = \sum_{n=0}^{\infty} a_n x^n \text{ making } y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \]

Substituting, \( y' \) and \( y'' \) into the differential equation produces:

\[ \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2 \sum_{n=1}^{\infty} n a_n x^n - 4 \sum_{n=0}^{\infty} a_n x^n = 0 \]

Shifting the first and second series so that they each have an \( x^n \) factor.

\[ \sum_{n=1}^{\infty} (n+1)(n)a_{n+1} x^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - 2 \sum_{n=1}^{\infty} n a_n x^n - 4 \sum_{n=0}^{\infty} a_n x^n = 0 \]

Adding the \( n = 0 \) term in the second and fourth series so that all series start with an index of \( n = 1 \) and condensing into one series give:

\[ 2a_2 - 4a_0 + \sum_{n=1}^{\infty} \left( (n+1)(n)a_{n+1} + (n+2)(n+1)a_{n+2} - 2n a_n - 4a_n \right) x^n = 0 \]

\[ 2a_2 - 4a_0 = 0 \quad (n+1)(n)a_{n+1} + (n+2)(n+1)a_{n+2} - 2n a_n - 4a_n = 0 \]

\[ a_2 = 2a_0 \quad a_{n+2} = \frac{2(n+2)a_n - n(n+1)a_{n+1}}{(n+2)(n+1)} \]

So our recurrence relation simplifies to

\[ a_{n+2} = \frac{2}{n+1} a_n - \frac{n}{n+2} a_{n+1} \]
As of now our solution is:

\[ y = a_0 + a_1 x + 2a_0 x^2 + \left( a_1 - \frac{2}{3} a_0 \right) x^3 + \left( \frac{5}{3} a_0 - \frac{1}{2} a_1 \right) x^4 + \left( a_1 - \frac{4}{3} a_0 \right) x^5 + \left( \frac{14}{9} a_0 - \frac{12}{15} a_1 \right) x^6 \]

Or

\[ y = a_0 \left( 1 + 2x^2 - \frac{2}{3} x^3 + \frac{5}{3} x^4 - \frac{4}{3} x^5 + \frac{14}{9} x^6 \right) + a_1 \left( x + x^3 - \frac{1}{2} x^4 + x^5 - \frac{4}{5} x^6 \right) \]

This solution does not look like the series expansion for \( e^{2x} \) that we are expecting. Noticing the first and third terms in the first linearly independent solution: \( 1 + 2x^2 \) match the first and third terms in the series but we seem to be missing the second term: \( 2x \). I will rewrite the linear term in the second linearly independent solution so that it has a \( 2x \) term in it. I will do this by letting

\[ a_1 = 2a_0 + K \]

Making

\[ a_3 = a_1 - \frac{1}{3} a_2 = \frac{4}{3} a_0 + K \quad a_4 = \frac{2}{3} a_2 - \frac{1}{2} a_3 = \frac{2}{3} a_0 - \frac{1}{2} K \]

\[ a_5 = a_1 - \frac{4}{3} a_0 = \frac{4}{15} a_0 + \frac{4}{5} K \]

Now our solution is

\[ y = a_0 + (2a_0 + K) x + 2a_0 x^2 + \left( \frac{4}{3} a_0 + K \right) x^3 + \left( \frac{2}{3} a_0 - \frac{1}{2} K \right) x^4 + \left( \frac{4}{15} a_0 + \frac{4}{5} K \right) x^5 \]

Separating this into two linearly independent solutions gives

\[ y = a_0 \left( 1 + 2x + 2x^2 + \frac{4}{3} x^3 + \frac{2}{3} x^4 + \frac{4}{15} x^5 \right) + K \left( x + x^3 - \frac{1}{2} x^4 + \frac{4}{5} x^5 \right) \]

Now the first linearly independent solution has the expansion for \( e^{2x} \) so our solution simplifies to

\[ y = a_0 e^{2x} + K \left( x + x^3 - \frac{1}{2} x^4 + \frac{4}{5} x^5 \right) \]

Now that we have one solution we normally would apply the reduction of order algorithm to find the second solution but the integral it produces is quite unpleasant so we must accept the series expansion as our second linearly independent solution.

400. Find the first six terms of the solution to the differential equation:

\[ y'' + xy' + y = 0 \]
401. Find the first six terms of the solution to the differential equation:

\[(x + 1)y'' - xy' - y = 0\]

402. Find the first six terms of the solution to the differential equation:

\[y'' + (3x - 1)y' - y = 0\]

403. Find the first six terms of the solution to the differential equation:

\[(2x + 3)y'' - (4x + 8)y' + 4y = 0\]

404. Find the first six terms of the solution to the differential equation:

\[(3x + 4)y'' - (3x + 4)y' + (2x + 3)y = 0\]

405. Find the first six terms of the solution to the differential equation:

\[(x^2 - 2x)y'' + (2 - x^2)y' + (2x - 2)y = 0\]

406. Find the first six terms of the solution to the differential equation:

\[y'' - \frac{x}{x - 1}y' + \frac{1}{x - 1}y = 0\]

407. Find the first six terms of the solution to the differential equation:

\[4xy'' + 3y' + xy = 0\]

408. Find the first six terms of the solution to the differential equation:

\[(x^2 + 1)y'' + y = 0\]

409. Find the first six terms of the solution to the differential equation:

\[(2x + 1)y'' - 2y' - (2x + 3)y = 0\]

410. Find the first six terms of the solution to the differential equation:

\[(x^2 + 2)y'' + xy' - y = 0\]
411. Find the first 5 terms in \( y_1 \) in the series expansion about \( x = 0 \) for a solution to:

\[
y'' - 2xy' - 2y = 0
\]

412. Find the first 5 terms in \( y_1 \) in the series expansion about \( x = 0 \) for a solution to:

\[
y'' + xy' - (4 + 2x)y = 0
\]

413. Show that the solutions to

\[
y'' = y' + y
\]

is given by

\[
\sum_{n=0}^{\infty} \frac{F_n x^n}{n!}
\]

where \( F_n \) is the \( n \)th Fibonacci number.

414. Use the power series for \( \sin(x) \), \( \cos(x) \) and \( e^x \) to derive Euler’s Equation:

\[
e^{ix} = \cos(x) + i\sin(x)
\]

415. Use your knowledge of power series to evaluate

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1}
\]

### 3.2 Method of Frobenius:

We will now try to find a power series solution to the differential equation:

\[
y'' + P(x)y' + Q(x)y = 0
\]

centered at a point where either \( P(x) \) or \( Q(x) \) is not analytic. To motivate our procedure let’s reconsider the Cauchy Euler equation.

\[
x^2 \frac{d^2 y}{dx^2} + ax \frac{dy}{dx} + by = 0
\]

After dividing by \( x^2 \) we get

\[
\frac{d^2 y}{dx^2} + a \frac{dy}{dx} + \frac{b}{x^2} y = 0
\]

So both
3.2. METHOD OF FROBENIUS:

\[ P(x) = \frac{a}{x} \quad \text{and} \quad Q(x) = \frac{b}{x^2} \]

are not analytic at \( x = 0 \).

The Cauchy Euler equation has a characteristic equation

\[ r(r - 1) + ar + b = 0 \]

which can be created by the following equation

\[ r(r - 1) + xP(x) + x^2Q(x) = 0 \]

This equation is quite similar to the characteristic equation we use in the method of Frobenius.

First a definition:

\( x = x_0 \) is a Singular Point of the above differential equation if either \( P(x) \) or \( Q(x) \) is not analytic at \( x = x_0 \). \( x = x_0 \) is a Regular Singular Point if the following limits exist:

\[ p_0 = \lim_{x \to x_0} \frac{(x - x_0)P(x)}{x} \quad q_0 = \lim_{x \to x_0} \frac{(x - x_0)^2Q(x)}{x} \]

To find a series solution about a regular singular point we use the Method of Frobenius. In the Method of Frobenius we look for a series solution of the form:

\[ y = \sum_{n=0}^{\infty} a_n x^{n+r} \quad \text{making} \quad y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} \]

The values of \( r \) that we will use are the roots of the Indicial Equation:

\[ r(r - 1) + p_0r + q_0 = 0 \]

There will be three cases we will have to consider based on the roots: \( r_1 \) and \( r_2 \) to the indicial equation. Note: we always take \( r_1 \) to be the larger of the two roots. The first case we will consider is the case when the roots differ by a non integer. That is:

\[ r_1 - r_2 \notin \mathbb{Z} \]

An Example:

Find the first 6 terms in the series expansion for the solution to the following differential equation centered at \( x = 0 \).

\[ 2x(x - 1)y'' + 3(x - 1)y' - y = 0 \]

In standard form this equation is:

\[ y'' + \frac{3}{2x} y' + \frac{-1}{2x(x - 1)} y = 0 \]

\[ p_0 = \lim_{x \to 0} x \frac{3}{2x} = \frac{3}{2} \quad q_0 = \lim_{x \to 0} x^2 \frac{-1}{2x(x - 1)} = 0 \]

The indicial equation is:
The roots are $r_1 = 0$ and $r_2 = -\frac{1}{2}$.

Inserting:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \text{ making } y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

into the original differential equation gives:

$$2 \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r} - 2 \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-1} + 3 \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} - 3 \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Shifting the first, third and fifth summation so that all series will have an $x^{n+r-1}$ factor:

$$2 \sum_{n=1}^{\infty} (n+r-1)(n+r-2)a_{n-1} x^{n+r-1} - 2 \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-1} + 3 \sum_{n=1}^{\infty} (n+r-1)a_n x^{n+r-1} - 3 \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

Adding the $n = 0$ terms in the second and forth series and condensing into a single series gives:

$$(-2(r-1)a_0 - 3ra_0)x^{r-1} + \sum_{n=1}^{\infty} \left(2(n+r-1)(n+r-2)a_{n-1} - 2(n+r)(n+r-1)a_n + 3(n+r-1)a_{n-1} - 3(n+r)a_n - a_{n-1}\right)x^{n+r-1} = 0$$

Setting the coefficients of the polynomial on the left hand side equal to zero: the coefficients of the polynomial on the right hand side. We see that the coefficient of $x^{r-1}$ is zero at both $r_1 = 0$ and $r_2 = -\frac{1}{2}$ so $a_0$ is a free is not necessarily zero. We also get our recurrence relation from setting the coefficient of $x^{n+r-1}$ to zero.

$$2(n+r-1)(n+r-2)a_{n-1} - 2(n+r)(n+r-1)a_n + 3(n+r-1)a_{n-1} - 3(n+r)a_n - a_{n-1} = 0$$

Solving for $a_n$

$$a_n = a_{n-1} \frac{1-3(n+r-1) - 2(n+r-1)(n+r-2)}{-2(n+r)(n+r-1) - 3(n+r)}$$

We will find two linearly independent solutions by substituting our two values of $r$ into the recurrence relation. For $r_1 = 0$

$$a_n = a_{n-1} \frac{1-3(n-1) - 2(n-1)(n-2)}{-2n(n-1) - 3n}$$

Which simplifies to:

$$a_n = \left(\frac{2n-3}{2n+1}\right)a_{n-1}$$

To obtain the first 6 terms in the solution we will find the first 3 terms in $y_1$ using the above recurrence relation created by using $r_1 = 0$ and the first 3 terms in $y_2$ using the using a recurrence relation created by using $r_2 = -\frac{1}{2}$. The first 3 terms in this recurrence relation will have coefficients $a_0$, $a_1$ and $a_2$. They are:
3.2. METHOD OF FROBENIUS:

\[ a_1 = -\frac{1}{3}a_0 \quad a_2 = \frac{1}{5}a_1 = \frac{1}{5} \left( -\frac{1}{3}a_0 \right) = -\frac{1}{15}a_0 \]

Using \( r_1 = 0 \) the first 3 terms in \( y_1 \) are:

\[ y_1 = a_0 + a_1x + a_2x^2 \]

\[ y_1 = a_0 \left( 1 - \frac{x}{3} - \frac{x^2}{15} \right) \]

Since this is a second order differential equation we would expect the form of the solution to be:

\[ y = C_1y_1 + C_2y_2 \]

In our answer \( C_1 \) is the constant \( a_0 \).

To find \( y_2 \) we insert \( r_2 = -\frac{1}{2} \) into the recurrence relation:

\[ a_n = \frac{1 - 3(n - \frac{3}{2}) - 2(n - \frac{3}{2})(n - \frac{5}{2})}{-2(n - \frac{1}{2})(n - \frac{5}{2}) - 3(n - \frac{1}{2})} \]

After a bit of simplifying we arrive at our recurrence relation:

\[ a_n = \left( \frac{2n - 1}{n} \right) a_{n-1} \]

The first three terms in this recurrence relation will have coefficients \( a_0, a_1 \) and \( a_2 \). They are:

\[ a_1 = a_0 \quad a_2 = \frac{3}{2}a_1 = \frac{3}{2}a_0 \]

\[ y_2 = a_0x^{-\frac{1}{2}} + a_1x^{\frac{1}{2}} + a_2x^{\frac{3}{2}} \]

\[ y_2 = a_0x^{-\frac{1}{2}} + a_0x^{\frac{1}{2}} + \frac{3}{2}a_0x^{\frac{3}{2}} \]

\[ y_2 = a_0 \left( x^{-\frac{1}{2}} + x^{\frac{1}{2}} + \frac{3}{2}x^{\frac{3}{2}} \right) \]

An Example:

Solve the following differential equation by method of Forbenious.

\[ 4xy'' + 2y' + y = 0 \]

In standard form our equation is:

\[ y'' + \frac{1}{2x}y' + \frac{1}{4x}y = 0 \]

\[ p_0 = \lim_{x \to 0} x \left( \frac{1}{2x} \right) = \frac{1}{2} \]
Our indicial equation becomes:

\[ r(r - 1) + \frac{1}{2} r = 0 \]

Which has roots \( r_1 = \frac{1}{2} \) and \( r_2 = 0 \)

Assume a solution to the differential equation of the form:

\[ y = \sum_{n=0}^{\infty} a_n x^{n+r} \]

Therefore:

\[ y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} \]

Inserting the series for \( y, y' \) and \( y'' \) into our differential equation yields the following:

\[ \sum_{n=0}^{\infty} 4(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \]

Shifting the last series on the LHS of the above equation yields:

\[ \sum_{n=0}^{\infty} 4(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r)a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0 \]

Adding the \( n = 0 \) terms in the first two series and then combining the rest into one series yields:

\[ (4r(r-1)a_0 + 2ra_0)x^{r-1} + \sum_{n=1}^{\infty} [4(n+r)(n+r-1)a_n + 2(n+r)a_n + a_{n-1}]x^{n+r-1} = 0 \]

This gives the recurrence relation:

\[ 4(n+r)(n+r-1)a_n + 2(n+r)a_n + a_{n-1} = 0 \]

Which simplifies to:

\[ a_n = \frac{-a_{n-1}}{(2n+2r)(2n+2r-1)} \]

For \( r_1 = \frac{1}{2} \) our recurrence relation becomes:

\[ a_n = \frac{-a_{n-1}}{(2n+1)(2n)} \]

Substituting values of \( n \) into our recurrence relation yields:

\[ a_1 = \frac{-a_0}{3 \cdot 2} \quad a_2 = \frac{-a_1}{5 \cdot 4} = \frac{a_0}{5 \cdot 4 \cdot 3 \cdot 2} \]

\[ a_3 = \frac{-a_2}{7 \cdot 6} = \frac{-a_0}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} \]
The solution to this recurrence relation is:
\[ a_n = \frac{(-1)^n \cdot a_0}{(2n + 1)!} \]

Making the solution:
\[ y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} x^{n+1/2} = \sin(\sqrt{x}) \]

For \( r_2 = 0 \) our recurrence relation becomes:
\[ a_n = \frac{-a_{n-1}}{(2n)(2n - 1)} \]

Substituting values of \( n \) into our recurrence relation yields:
\[
\begin{align*}
a_1 &= \frac{-a_0}{2 \cdot 1} = \frac{-a_0}{2} \\
a_2 &= \frac{-a_1}{4 \cdot 3} = \frac{-a_0}{4 \cdot 3 \cdot 2} \\
a_3 &= \frac{-a_2}{6 \cdot 5} = \frac{-a_0}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}
\end{align*}
\]

The solution to this recurrence relation is:
\[ a_n = \frac{(-1)^n \cdot a_0}{(2n)!} \]

Making the solution:
\[ y_2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n = \cos(\sqrt{x}) \]

Determine if \( x = 0 \) is a regular or irregular singular point. If \( x = 0 \) is a regular singular point find the indicial equation and its roots.
\[ y'' + \left( \frac{e^{2x} - 2x - 1}{xe^x - x} \right) y' + \left( \frac{\sin(4x)}{(x + \sin(x))^2} \right) y = 0 \]

Determine if \( x = 0 \) is a regular or irregular singular point. If \( x = 0 \) is a regular singular point find the indicial equation and its roots.
\[ y'' + x \sin \left( \frac{2}{x} \right) y' + \left( \frac{\arctan(x)}{x} \right) y = 0 \]

Determine if \( x = 0 \) is a regular or irregular singular point. If \( x = 0 \) is a regular singular point find the indicial equation and its roots.
\[ y'' + \left( \frac{1}{x} - \frac{1}{\sin(x)} \right) y' + \left( \frac{\arcsin(x)}{\ln(x + 1)} \right) y = 0 \]

Solve by method of Frobenious. You should be able to recognise a pattern and identify the power series as a known function.
Solve by method of Frobenious. You should be able to recognise a pattern and identify the power series as a known function.

\[ y'' + \frac{2}{x} y' + y = 0 \]

Find the first 5 terms in \( y_1 \) in the series expansion about \( x = 0 \) for a solution to:

\[ x^2 y'' - xy' + (1 - x)y = 0 \]

Find the first 5 terms in \( y_1 \) in the series expansion about \( x = 0 \) for a solution to:

\[ (x^2 - 2x)y'' + (2 - x^2)y' + (2x - 2)y = 0 \]

Find the first 5 terms in \( y_1 \) in the series expansion about \( x = 0 \) for a solution to:

\[ (6x^2 + 2x)y'' + (x + 1)y' - y = 0 \]

Find the first 5 terms in \( y_1 \) in the series expansion about \( x = 0 \) for a solution to:

\[ xy'' + y' + xy = 0 \]

Find the first 5 terms in \( y_1 \) in the series expansion about \( x = 0 \) for a solution to:

\[ xy'' + y' - 4y = 0 \]

Find the first 5 terms in \( y_1 \) in the series expansion about \( x = 0 \) for a solution to:

\[ xy'' + 2y' + xy = 0 \]

Show that one solution to

\[ x^2 y'' = (a^2 - a + bx)y \]

is

\[ y = x^a \left( 1 + \frac{bx}{1! \cdot 2a} + \frac{(bx)^2}{2! \cdot (2a)(2a + 1)} + \frac{(bx)^3}{3! \cdot (2a)(2a + 1)(2a + 2)} + \ldots \right) \]
3.3 The Gamma Function

As you have seen many of our series solutions involve factorials. The problem factorials is that they are only defined for non-negative integers. The gamma function extend the idea of factorials to all positive real numbers. The Gamma Function is given by the improper integral:

\[ \Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt \quad x > 0 \]

The first important property that we will prove of the gamma function shows its similarities to the factorial function:

**Thm:**

\[ \Gamma(x + 1) = x\Gamma(x) \]

**Proof:**

\[ \Gamma(x + 1) = \int_0^\infty e^{-t}t^{x}dt \]

Using integration by parts

\[ u = t^x \quad dv = e^{-t}dt \quad du = xt^{x-1}dt \quad v = -e^{-t} \]

\[ \Gamma(x + 1) = \lim_{b \to \infty} \left[ -t^xe^{-t} \right]_0^b + x \int_0^b t^{x-1}e^{-t}dt \]

The first term is zero at both zero and infinity so we have

\[ \Gamma(x + 1) = x \int_0^\infty e^{-t}t^{x-1}dt = x\Gamma(x) \]

This theorem demonstrates the relationship with the factorial function. The next result is a direct corollary to the above theorem and is the most well known property of the gamma function.

If \( n \) is a positive integer

\[ \Gamma(n) = (n - 1)! \]

**Example:**

Calculate \( \Gamma\left(\frac{1}{2}\right) \)

\[ \Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t}t^{\frac{1}{2}}dt = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}dt \]

Substituting

\[ u = \sqrt{t} \quad 2du = \frac{dt}{\sqrt{t}} \quad \text{when } t = 0 \quad u = 0 \quad \text{as } t \to \infty \quad u \to \infty \]

\[ \Gamma\left(\frac{1}{2}\right) = 2 \int_0^\infty e^{-u^2}du \]
Although we cannot find an antiderivative for \( e^{-u^2} \) we can evaluate this integral with a clever trick

\[
\Gamma\left(\frac{1}{2}\right) = 2 \int_0^\infty e^{-u^2} \, du = 2 \int_0^\infty e^{-v^2} \, dv
\]

So

\[
\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \left(2 \int_0^\infty e^{-u^2} \, du\right) \cdot \left(2 \int_0^\infty e^{-v^2} \, dv\right) = 4 \int_0^\infty \int_0^\infty e^{-u^2-v^2} \, du \, dv
\]

Converting to polar coordinates

\[
\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4 \int_0^{\pi/2} \int_0^\infty e^{-r^2} r \, dr \, d\theta
\]

\[
\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = -2 \int_0^{\pi/2} \lim_{b \to \infty} e^{-r^2} \bigg|_b^0 \, d\theta = 2 \int_0^{\pi/2} d\theta = \pi
\]

So

\[
\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}
\]

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Find

\[
\Gamma\left(\frac{3}{2}\right)
\]

429.

Show

\[
\lim_{x \to 0^+} \Gamma(x) = \infty
\]

430.

Use the principle of mathematical induction to prove:

\[
\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{4^n n!} \quad n = 0, 1, 2, ...
\]

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Show that the definition of the Gamma function is equivalent to Euler’s original definition:

\[
\Gamma(x) = \int_0^1 \left( \ln\left(\frac{1}{t}\right) \right)^{x-1} \, dt
\]

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Another important function related to the Gamma function is Euler’s Psi function: the derivative of the logarithm of the Gamma function.

\[
\psi(x) = \frac{d}{dx} \ln\left(\Gamma(x)\right) = \frac{\Gamma'(x)}{\Gamma(x)}
\]
Show the following property of the Psi function

\[ \psi(x + 1) = \frac{1}{x} + \psi(x) \]

Use the above results to also show for positive integers \( n \)

\[ \psi(n + 1) = \psi(1) + \sum_{k=1}^{n} \frac{1}{k} \]

### 3.4 Bessel’s Equation

We will now consider a differential equation of great importance in applied mathematics. The Bessel Equation of order \( v \) is:

\[ x^2 y'' + xy' + (x^2 - v^2)y = 0 \]

or in standard form:

\[ y'' + \frac{1}{x} y' + \left(1 - \frac{v^2}{x^2}\right)y = 0 \]

We see that \( x = 0 \) is a singular point of the Bessel Equation. Since both

\[ P_0 = \lim_{x \to 0} \frac{x}{x} = 1 \quad \text{And} \quad Q_0 = \lim_{x \to 0} x^2 \left(1 - \frac{v^2}{x^2}\right) = -v^2 \]

exist \( x = 0 \) is a regular singular point and our indicial equation is

\[ r(r - 1) + r - v^2 = 0 \quad \text{Or} \quad r^2 - v^2 = 0 \]

Which has roots

\[ r = \pm v \]

If the difference between the roots: \( 2n \) is not an integer then the method of Frobenius gives two linearly independent solutions:

\[ J_v = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+v}}{2^{2n+v} n! \Gamma(1 + v + n)} \]

and

\[ J_{-v} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-v}}{2^{2n-v} n! \Gamma(1 - v + n)} \]

An Example:

Solve the Bessel equation of order \( \frac{1}{2} \)

\[ x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) = 0 \]
In standard form we have

\[ y'' + \frac{1}{x} y' + \left(1 - \frac{1}{4x^2}\right)y = 0 \]

We see that \( x = 0 \) is a singular point. To show it is a regular singular point we compute:

\[ P_0 = \lim_{x \to 0} \frac{1}{x} = 1 \quad Q_0 = \lim_{x \to 0} x^2 \left(1 - \frac{1}{4x^2}\right) = -\frac{1}{4} \]

Since both limits exist we see that \( x = 0 \) is a regular singular point and we can use the method of Frobenius to find a series solution about \( x = 0 \). The indicial equation and roots are:

\[ r(r - 1) + r - \frac{1}{4} = 0 \quad r = \pm \frac{1}{2} \]

\[ y = \sum_{n=0}^{\infty} a_n x^{n+r} \quad y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1} \quad y = \sum_{n=0}^{\infty} (n + r)(n + r - 1) a_n x^{n+r-2} \]

Inserting the series for \( y, y' \) and \( y'' \) into the original differential equation gives:

\[ \sum_{n=0}^{\infty} (n + r)(n + r - 1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n + r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{n+r} = 0 \]

Shifting the third series so that it too has an exponent of \( n + r \)

\[ \sum_{n=0}^{\infty} (n + r)(n + r - 1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n + r)a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} - \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{n+r} = 0 \]

Adding the first and second terms in the first, second and fourth series will produce four series all with an index starting at \( n = 2 \) that can be simplified into a single series.

\[ \left( r(r - 1) + r - \frac{1}{4}\right) a_0 x^r + \left( (1 + r)r + (1 + r) - \frac{1}{4}\right) a_1 x^{1+r} + \sum_{n=2}^{\infty} \left( (n + r)(n + r - 1) + (n + r) - \frac{1}{4}\right) a_n + a_{n-2} x^{n+r} = 0 \]

The coefficient of \( a_0 x^r \) is identical to the indicial equation and is zero at both values of \( r \) making \( a_0 \) not zero whereas the coefficient of \( a_1 x^{1+r} \) is not zero at both values of \( r \) making \( a_1 = 0 \). Our recurrence relation is:

\[ \left( (n + r)(n + r - 1) + (n + r) - \frac{1}{4}\right) a_n + a_{n-2} = 0 \]

\[ a_n = \frac{-a_{n-2}}{(n + r)^2 - \frac{1}{4}} \]

For the larger root \( r = \frac{1}{2} \) our recurrence relation simplifies to:

\[ a_n = \frac{-a_{n-2}}{n(n + 1)} \]

Producing a sequence of terms in \( a_n \)

\[ a_2 = \frac{-a_0}{2} \cdot \frac{-a_0}{3!} = \frac{-a_0}{3!} \]

\[ a_3 = \frac{-a_1}{4} \cdot \frac{3}{3} = 0 \]

Since \( a_5 \) depends on \( a_3 \) it too must be zero. In fact \( a_n = 0 \) for all odd values of \( n \)
\[ a_4 = \frac{-a_2}{5 \cdot 4} = \frac{a_0}{5} \cdot \frac{3}{2} = \frac{a_0}{5!} \]  

\[ a_6 = \frac{-a_4}{7 \cdot 6} = \frac{-a_0}{7!} \]

Recognizing the pattern we see

\[ a_{2n} = \frac{(-1)^n a_0}{(2n + 1)!} \]

And the first solution is

\[ y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n a_0}{(2n + 1)!} \frac{x^{2n+1}}{\sqrt{x}} = \frac{1}{\sqrt{x}} \sum_{n=0}^{\infty} \frac{(-1)^n a_0}{(2n + 1)!} x^{2n+1} \]

Recognizing the series gives:

\[ y_1 = J_{-\frac{1}{2}}(x) = a_0 \frac{\cos(x)}{\sqrt{x}} \]

Since \( r_1 - r_2 \) is an integer the second linearly independent solution would be quite difficult to find using series, but we were able to find a closed form expression for \( y_1 = \frac{\cos(x)}{\sqrt{x}} \) so we can use the method of reduction of order form chapter 2 to find the second linearly independent solution.

\[ y'' + \frac{1}{x} y' + \left(1 - \frac{1}{4x^2}\right) y = 0 \]

\[ v = \int \frac{e^{-\frac{1}{2}dx}}{\cos^2(x)} \frac{dx}{x} = \int \sec^2(x)dx = \tan(x) \]

Our second linearly independent solution is

\[ y_2 = y_1 v = \frac{\cos(x)}{\sqrt{x}} \tan(x) \]

\[ y_2 = J_{\frac{1}{2}}(x) = \sin(x) \frac{1}{\sqrt{x}} \]

And the homogenous solution is:

\[ y = C_1 \cos(x) \frac{1}{\sqrt{x}} + C_2 \sin(x) \frac{1}{\sqrt{x}} \]

A few important properties of the solutions to the Bessel equation are:

\[ \frac{d}{dx} \left(x^v J_v(x)\right) = x^v J_{v-1}(x) \quad \quad \frac{d}{dx} \left(x^{-v} J_v(x)\right) = -x^{-v} J_{v+1}(x) \]

\[ J_{v+1}(x) = \frac{2v}{x} J_v(x) - J_{v-1}(x) \quad \quad J_{v+1}(x) = J_{v-1}(x) - 2J'_v(x) \]

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The Bessel functions of order \( n + \frac{1}{2} \) for integer \( n \) are related to the spherical Bessel functions. Use one of the above properties of the Bessel function and the results of the example to calculate \( J_{\frac{1}{2}} \) and \( J_{\frac{3}{2}} \).
434. Show the orthogonality of the Bessel functions by showing
\[
\int_0^1 xJ_{-\frac{1}{2}}J_{\frac{1}{2}} dx = 0
\]

435. Show that the substitution \( z = \sqrt{x}y \) transforms the Bessel equation to normal form:
\[
z'' + \left( 1 + \frac{1 - 4v^2}{4x^2} \right) z = 0
\]

436. The parametric Bessel equation is:
\[
x^2 y'' + xy' + (\lambda^2 x^2 - v^2)y = 0 \quad x > 0
\]
Show
\[
y = C_1 J_v(\lambda x) + C_2 J_{-v}(\lambda x) \quad v \notin \mathbb{Z}
\]
is the solution

437. Use the results from the previous problem and to find the solution to
\[
x^2 y'' + xy' + \left( 25x^2 - \frac{1}{4} \right) y = 0 \quad x > 0
\]

438. Use ideas from the previous two problems and the fact \( i^2 x^2 = -x^2 \) to solve
\[
x^2 y'' + xy' - \left( x^2 + \frac{1}{4} \right) y = 0 \quad x > 0
\]

439. Use the change of variable \( y = x^{-\frac{1}{2}} v(x) \) to solve
\[
x^2 y'' + 2xy' + \lambda^2 x^2 y = 0
\]
Chapter 4

Laplace Transform

4.1 Calculating Laplace and Inverse Laplace Transforms

The Laplace Transform of a function \( f(x) \) will be a function of \( s \) given by the improper integral:

\[
\mathcal{L}(f) = \int_0^\infty f(x)e^{-sx}dx
\]

provided the integral converges. Functions that outgrow \( e^{kx} \), for constant \( k \), do not have a Laplace Transform since the integral diverges. So \( f(x) = e^{nx} \) will not have a Laplace Transform for \( n > 1 \). Let us now try to find the Laplace Transform of \( f(x) = e^{ax} \).

\[
\mathcal{L}(e^{ax}) = \int_0^\infty e^{ax}e^{-sx}dx = \int_0^\infty e^{(a-s)x}dx
\]

\[
= \lim_{b \to \infty} \left. \frac{e^{(a-s)x}}{a-s} \right|_0^b = \lim_{b \to \infty} \frac{e^{(a-s)b}}{a-s} - \frac{1}{a-s} = \frac{1}{s-a}
\]

For \( s > a \).

Here is a basic table of Laplace transforms:
CHAPTER 4. LAPLACE TRANSFORM

\[ f(x) = e^{ax} \]

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( \mathcal{L}(f(x)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^{ax} )</td>
<td>( \frac{n}{s-a} )</td>
</tr>
<tr>
<td>( \sin(ax) )</td>
<td>( \frac{a}{s^2+a^2} )</td>
</tr>
<tr>
<td>( \cos(ax) )</td>
<td>( \frac{s}{s^2+a^2} )</td>
</tr>
<tr>
<td>( x^n ), ( n \in \mathbb{N} )</td>
<td>( \frac{n!}{s^{n+1}} )</td>
</tr>
<tr>
<td>( e^{ax} - e^{bx} )</td>
<td>( \frac{a-b}{(s-a)(s-b)} )</td>
</tr>
<tr>
<td>( e^{ax} \sin(bx) )</td>
<td>( \frac{b}{(s-a)^2+b^2} )</td>
</tr>
<tr>
<td>( e^{ax} \cos(bx) )</td>
<td>( \frac{s-a}{(s-a)^2+b^2} )</td>
</tr>
<tr>
<td>( x \sin(ax) )</td>
<td>( \frac{2a}{(s^2+a^2)^2} )</td>
</tr>
<tr>
<td>( x \cos(ax) )</td>
<td>( \frac{s^2-a^2}{(s^2+a^2)^2} )</td>
</tr>
<tr>
<td>( \sinh(ax) )</td>
<td>( \frac{s}{s^2+a^2} )</td>
</tr>
<tr>
<td>( \cosh(ax) )</td>
<td>( \frac{a}{s^2-a^2} )</td>
</tr>
<tr>
<td>( x^r ), ( r \in \mathbb{R} ), ( r &gt; -1 )</td>
<td>( \frac{1}{(r+1)!} )</td>
</tr>
<tr>
<td>( \frac{1}{\sqrt{x}} )</td>
<td>( \frac{\sqrt{\pi}}{2a^{\frac{3}{2}}} )</td>
</tr>
<tr>
<td>( \sqrt{x} )</td>
<td>( \frac{\sqrt{\pi}}{2a^{\frac{1}{2}}} )</td>
</tr>
<tr>
<td>( x^{n-\frac{1}{2}} ), ( n \in \mathbb{N} )</td>
<td>( \frac{1-35...((2n-1)\sqrt{\pi})}{2^{n+\frac{1}{2}} s^{n+\frac{1}{2}}} )</td>
</tr>
<tr>
<td>( \sin(ax) \cosh(ax) - \cos(ax) \sinh(ax) )</td>
<td>( \frac{4a^2}{s^{4}+4a^4} )</td>
</tr>
<tr>
<td>( \sin(ax) \sinh(ax) )</td>
<td>( \frac{2a^2}{s^4+4a^4} )</td>
</tr>
<tr>
<td>( \sinh(ax) - \sin(ax) )</td>
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</tr>
<tr>
<td>( \cosh(ax) - \cos(ax) )</td>
<td>( \frac{2a^2}{s^4-a^4} )</td>
</tr>
<tr>
<td>( \sin(ax) - ax \cos(ax) )</td>
<td>( \frac{2a^4}{(s^2+a^2)^2} )</td>
</tr>
<tr>
<td>( \sin(ax) + ax \cos(ax) )</td>
<td>( \frac{2a^4}{(s^2+a^2)^2} )</td>
</tr>
<tr>
<td>( x^3 \sin(ax) )</td>
<td>( \frac{-2a(a^2-3a^2)}{(s^2+a^2)^3} )</td>
</tr>
<tr>
<td>( x^3 \cos(ax) )</td>
<td>( \frac{2a(s^2-3a^2)}{(s^2+a^2)^3} )</td>
</tr>
<tr>
<td>( e^{ax}g(x) )</td>
<td>( G(s-a) )</td>
</tr>
<tr>
<td>( x^n g(x) )</td>
<td>( \left( -1 \right)^n \frac{d^n}{ds^n} \mathcal{L}(g(x)) )</td>
</tr>
</tbody>
</table>

Notice there is not a Laplace Transform for any function that outgrows \( f(x) = e^{ax} \). This is because the improper integral from the definition will diverge.

One key property of the Laplace Transform is that it is a Linear Transformation, meaning:

\[ \mathcal{L}(af(x) + bg(x)) = a\mathcal{L}(f(x)) + b\mathcal{L}(g(x)) \]

**An Example:** Find the Laplace Transform of:

\[ f(x) = 4x \sin(3x) + x^2 e^{6x} \]

Using the table we see that the Laplace Transform of \( 4x \sin(3x) \) is \( 4 \cdot \frac{2a}{(s^2+a^2)^2} \), and the Laplace Transform of \( x^2 e^{6x} \) is \( \frac{2}{(s-6)^2} \). So:
\[ L(f(x)) = \frac{24s}{(s^2 + 9)^2} + \frac{2}{(s - 6)^3} \]

**An Example:** Find the inverse Laplace Transform of:

\[ F(s) = \frac{s^3 - 8s^2 + 23s - 7}{(s^2 - 4s + 20)(s - 3)^2} \]

Looking at the table of Laplace Transforms we notice that there is not an inverse Laplace Transform for a function with a repeated linear factor multiplied by a quadratic factor so we must use partial fraction decomposition to separate this function into a sum of functions that do have an inverse Laplace Transform. By partial fractions \[ F(s) = \frac{s - 3}{s^2 - 4s + 20} + \frac{1}{(s - 3)^2} \]

Looking at the table we notice all terms with a quadratic denominator are written in the completed square form. So we will complete the square on the quadratic denominator and do nothing to the second term since we can find the inverse Laplace Transform it as it is:

\[ F(s) = \frac{s - 3}{(s - 2)^2 + 16} + \frac{1}{(s - 3)^2} \]

To find the inverse Laplace of a term with a denominator: \((s - 2)^2 + 4^2\) we need to have either an \(s - 2\) or a 4 in the numerator so I will rewrite the \(s - 3\) term as \(s - 2 - 1\) and separate the fraction into two fractions:

\[ F(s) = \frac{s - 2}{(s - 2)^2 + 4^2} - \frac{1}{(s - 2)^2 + 4^2} + \frac{1}{(s - 3)^2} \]

I will now multiply the second term by \(\frac{4}{4}\) to make it fit the table:

\[ F(s) = \frac{s - 2}{(s - 2)^2 + 4^2} - \frac{1}{4} \left( \frac{4}{(s - 2)^2 + 4^2} \right) + \frac{1}{(s - 3)^2} \]

Using the table to find the inverse Laplace Transform gives:

\[ f(x) = e^{2x} \cos(4x) - \frac{1}{4} e^{2x} \sin(4x) + x e^{3x} \]

440. Find the Laplace Transform of:

\[ f(x) = x^3 - 3x \cos(4x) \]

441. Find the Laplace Transform of:

\[ f(x) = x^2 \sinh(3x) + e^{2x} \sin(3x) \]

442. Find the Laplace Transform of:

\[ f(x) = x^4 e^{5x} - \sin(2x) \sinh(2x) \]

443. Use trig identities and the table in this section to find the Laplace transform of:
$f(x) = \sin(Ax + B) \quad g(x) = \cos(Ax + B)$

444. Use the definition to find the Laplace Transform of:

$$f(x) = \begin{cases} 1 & x < 2 \\ x & 2 \leq x < 6 \\ x^2 & x \geq 6 \end{cases}$$

445. Find the Laplace Transform of:

$$f(x) = x^2 - x^5$$

446. Find the Inverse Laplace Transform of:

$$F(s) = \frac{3s}{s^2 + 4} + \frac{12}{s^2 - 10s + 34}$$

447. Find the Inverse Laplace Transform of:

$$F(s) = \frac{500 + 40s}{(s^2 + 25)^2}$$

448. Find the Inverse Laplace Transform of:

$$F(s) = \frac{4s^3 + 4s^2 - 27s - 18}{s^4 - 3s^3 + 2s^2}$$

449. Find the Inverse Laplace Transform of:

$$F(s) = \frac{3s^4 + s^3 - 5s^2 - 10s - 25}{s^5 + 2s^4 + 5s^3}$$

450. Find the Inverse Laplace Transform of:

$$F(s) = \frac{9s^2 - 30s + 49}{(s - 3)^2(s^2 + 1)}$$

451. Find the Inverse Laplace Transform of:

$$F(s) = \frac{3s^3 - 6s^2 + 39s + 54}{(s^2 - 4s + 13)(s^2 + 9)}$$

452. Find the Inverse Laplace Transform of:

$$F(s) = \sum_{n=1}^{\infty} \frac{1}{s^n}$$

You should be able to recognize the power series.

453. Find the Inverse Laplace Transform of:

$$F(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{s^{2n+2}}$$

You should be able to recognize the power series.
454. Find the Inverse Laplace Transform of:

\[ F(s) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{s^{2n+2}} \]

You should be able to recognize the power series.

455. Use the series expansion for \( \ln(1 + x) \) to find

\[ \mathcal{L}(\ln(1 + x)) \]

456. Show \( \mathcal{L}(e^{x^2}) \) does not exist.

457. Show \( \mathcal{L}\left(\frac{1}{x^2}\right) \) does not exist.

458. Use

\[
\cosh(x) = \frac{e^x + e^{-x}}{2} \quad \sinh(x) = \frac{e^x - e^{-x}}{2}
\]

to derive formulas for

\[ \mathcal{L}\{\cosh(ax + b)\} \quad \mathcal{L}\{\sinh(ax + b)\} \]

and

\[ \mathcal{L}\{\cosh^2(x)\} \quad \mathcal{L}\{\sinh^2(x)\} \]

Then find the inverse Laplace transform of

\[ \mathcal{L}\{\cosh^2(x)\} - \mathcal{L}\{\sinh^2(x)\} \]

What can you conclude about the value of \( \cosh^2(x) - \sinh^2(x) \)?

459. Calculate:

\[ \mathcal{L}(x^2 \cdot y') \]

460. Let

\[ y = \sum_{i=0}^{n} (1 + t)^n \]

and
\[ F(s) = \mathcal{L}(y) \]

Show that the coefficients of both \( \frac{1}{s^n} \) and \( \frac{1}{s^{n+1}} \) in \( F(s) \) are both \( n! \).

461.

Use the following property:

\[ \mathcal{L}\left( \frac{f(x)}{x} \right) = \int_s^\infty F(x) \, dx \]

to find

\[ \mathcal{L}\left( \frac{\sinh(x)}{x} \right) \]

462.

Use a power series expansion to show:

\[ \mathcal{L}^{-1}\left( \frac{e^{-\frac{x}{2}}}{\sqrt{s}} \right) = \frac{\cos(2\sqrt{t})}{\sqrt{\pi t}} \]

463.

Recall the Gamma function from the last chapter is given by

\[ \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt \]

Show that

\[ \mathcal{L}(x^r) = \frac{\Gamma(r + 1)}{s^{r+1}} \quad r > -1 \]

464.

Show

\[ \mathcal{L}\{y \cdot y'\} = \frac{1}{2} \left( s\mathcal{L}\{y^2\} - y^2(0) \right) \]

465.

Another interesting inverse Laplace Transform is:

\[ \mathcal{L}^{-1}\left( F(s) \right) = -\frac{1}{x} \mathcal{L}^{-1}\left( F'(s) \right) \]

Use this transform to calculate the inverse Laplace Transform of

\[ F(s) = \arctan\left( \frac{1}{s} \right) \]

466.

Use the previous problem to find the inverse Laplace Transform of

\[ F(s) = \ln(1 + s^2) \]
4.2 Solving Initial Value Problems

What is the use of Laplace Transforms in differential equations? The answer comes from the next derivation of the Laplace Transform of $y'(x)$:

\[
\mathcal{L}(y'(x)) = \int_0^\infty y'(x)e^{-sx}dx \quad \text{using integration by parts}
\]

\[
u = e^{-sx} \quad dv = y'(x)dx \quad du = -se^{-sx}dx \quad v = y(x)
\]

\[
\mathcal{L}(y'(x)) = \int_0^\infty y'(x)e^{-sx}dx = \lim_{b \to \infty} e^{-sb}y(b) - y(0) + s\mathcal{L}(y(x)) - sy(0) - y'(0)
\]

In a similar way we can find the Laplace Transform of $y''$, $y'''$ ...

\[
\mathcal{L}(y'(x)) = s\mathcal{L}(y(x)) - y(0)
\]

\[
\mathcal{L}(y''(x)) = s^2\mathcal{L}(y(x)) - sy(0) - y'(0)
\]

\[
\mathcal{L}(y'''(x)) = s^3\mathcal{L}(y(x)) - s^2y(0) - sy'(0) - y''(0)
\]

From the above equations it should be clear that you must have initial conditions evaluated at $x = 0$ to use Laplace Transforms to solve a differential equation.

The procedure to solve a differential equation using Laplace Transforms is: First, using the above tables take the Laplace Transform of both sides of the differential equation; Second, Solve for $\mathcal{L}(y(x))$ as a function of $s$; Third, using the above tables calculate the inverse Laplace of both sides of the equation.

A Example:

Solve:

\[
y'' + 4y = 20e^t \cos(t) \quad y(0) = 7 \quad y'(0) = 10
\]

Taking the Laplace transform of both sides gives:

\[
s^2\mathcal{L}(y) - 7s - 10 + 4\mathcal{L}(y) = 20 \frac{s - 1}{(s - 1)^2 + 1}
\]

Solving for $\mathcal{L}(y)$:

\[
\mathcal{L}(y)\left(s^2 + 4\right) = 7s + 10 + 20 \frac{s - 1}{(s - 1)^2 + 1}
\]
\[ L(y) = \frac{7s + 10}{s^2 + 4} + \frac{20s - 1}{((s - 1)^2 + 1)(s^2 + 4)} \]

Applying Partial Fractions to the second fraction on the right hand side and adding the results to the first fraction yields

\[ L(y) = \frac{4s - 2}{s^2 - 2s + 2} + \frac{3s + 4}{s^2 + 4} \]

Completing the square on the first fraction

\[ L(y) = \frac{4s - 2}{(s - 1)^2 + 1} + \frac{3s + 4}{s^2 + 4} \]

Manipulating the right hand side into our table of Laplace Transforms

\[ L(y) = \frac{s - 1}{(s - 1)^2 + 1} + \frac{2}{s^2 + 4} + \frac{2}{s^2 + 4} \]

Taking the inverse Laplace of each side

\[ y = 4e^t \cos(t) + 2e^t \sin(t) + 3 \cos(2t) + 2 \sin(2t) \]

An Example: Solve:

\[ y'' - 2y' - y = e^{2x} - e^x \quad y(0) = 1 \quad y'(0) = 3 \]

Taking the Laplace transform of both sides gives:

\[ s^2 L(y) - s - 3 - 2 \left( sL(y) - 1 \right) - L(y) = \frac{1}{s - 2} - \frac{1}{s - 1} \]

Solving for \( L(y) \):

\[ L(y)(s^2 - 2s - 1) = \frac{1}{s - 2} - \frac{1}{s - 1} + s + 1 \]

Solving for \( L(y) \) and factoring \( s^2 - 2s - 1 = (s - (1 + \sqrt{2}))(s - (1 - \sqrt{2})) \)

\[ L(y) = \frac{1}{(s - 2)(s - (1 + \sqrt{2}))(s - (1 - \sqrt{2}))} - \frac{1}{(s - 1)(s - (1 + \sqrt{2}))(s - (1 - \sqrt{2}))} + \frac{s + 1}{(s - (1 + \sqrt{2}))(s - (1 - \sqrt{2}))} \]

Expanding with partial fractions gives:

\[ L(y) = \frac{3(\sqrt{2} + 1)}{4} \frac{1}{s - (\sqrt{2} + 1)} - \frac{3(\sqrt{2} - 1)}{4} \frac{1}{s - (1 - \sqrt{2})} + \frac{1}{2} \frac{1}{s - 1} - \frac{1}{s - 2} \]

Calculating the inverse Laplace Transform

\[ y = \frac{3(\sqrt{2} + 1)}{4} e^{(\sqrt{2} + 1)x} - \frac{3(\sqrt{2} - 1)}{4} e^{(1 - \sqrt{2})x} + \frac{1}{2} e^x - e^{2x} \]

The Laplace Transform can also be used to solve linear differential equations without constant coefficients. To do so we would like to have nice formula for:
Unfortunately, these formulas either do not exist or are well beyond the scope of this book, so instead we will find formulas for the specific case $f(x) = x$. Let’s now calculate $\mathcal{L}(xy)$.

We know

$$\mathcal{L}(xf(x)) = -\frac{d}{ds}\mathcal{L}(f(x))$$

So

$$\mathcal{L}(xy) = -\frac{d}{ds}\mathcal{L}(y)$$

And

$$\mathcal{L}(xy') = -\frac{d}{ds}\mathcal{L}(y') = -\frac{d}{ds}\left(s\mathcal{L}(y) - y(0)\right) = -\mathcal{L}(y) - s\frac{d}{ds}\mathcal{L}(y)$$

And

$$\mathcal{L}(xy'') = -\frac{d}{ds}\mathcal{L}(y'') = -\frac{d}{ds}\left(s^2\mathcal{L}(y) - sy(0) - y'(0)\right) = -2s\mathcal{L}(y) - s^2\frac{d}{ds}\mathcal{L}(y) + y(0)$$

Using the notation $\mathcal{L}'(y) = \frac{d}{ds}\mathcal{L}(y)$ our formulas become:

$$\mathcal{L}(xy) = -\mathcal{L}'(y) \quad \mathcal{L}(xy') = -\mathcal{L}(y) - s\mathcal{L}'(y) \quad \mathcal{L}(xy'') = -2s\mathcal{L}(y) - s^2\mathcal{L}'(y) + y(0)$$

**A Example:**

Solve:

$$xy'' + 2(x - 1)y' - 2y = 0 \quad y(0) = 0$$

Taking the Laplace Transform of each side gives:

$$-2s\mathcal{L}(y) - s^2\mathcal{L}'(y) + y(0) - 2\mathcal{L}(y) - 2s\mathcal{L}'(y) - s\mathcal{L}(y) + y(0) - 2\mathcal{L}(y) = 0$$

Which simplifies to:

$$(s^2 + 2s)\mathcal{L}'(y) + (4s + 4)\mathcal{L}(y) = 0$$

This equation is now separable

$$\frac{\mathcal{L}'(y)}{\mathcal{L}(y)} = -\frac{4s + 4}{s(s + 2)}$$

Applying partial fraction to the right hand side gives.
\[ \frac{L'(y)}{L(y)} = -\frac{2}{s} - \frac{2}{s+2} \]

Integrating gives

\[ \ln \left( L(y) \right) = -2 \ln(s) - 2 \ln(s+2) + C \]

Treating \( C \) as \( \ln(K) \) and using some log rules we get:

\[ \ln(s^2(s+2)^2 L(y)) = \ln(K) \]

So

\[ L(y) = \frac{K}{s^2(s+2)^2} \]

Applying partial fraction to the right hand side gives

\[ L(y) = K \left( \frac{1}{4} \cdot \frac{1}{s} + \frac{1}{4} \cdot \frac{1}{s^2} + \frac{1}{4} \cdot \frac{1}{s+2} + \frac{1}{4} \cdot \frac{1}{(s+2)^2} \right) \]

Absorbing the \( \frac{1}{4} \) into the constant and calculating the inverse Laplace Transform gives:

\[ y = K \left( -1 + x + e^{-2x} + xe^{-2x} \right) \]

We see the first initial condition \( y(0) = 0 \) gives us no information about the constant \( K \) but our second initial condition \( y(1) = 2e^{-2} \) does. Applying this condition gives

\[ 2e^{-2} = K \left( -1 + 1 + e^{-2} + e^{-2} \right) \]

So \( K = 1 \) and the solution is

\[ y = \left( -1 + x + e^{-2x} + xe^{-2x} \right) \]

467. Solve:

\[ y' + y = e^x \quad y(0) = 1 \]

468. Solve:

\[ y' + 2y = \sin(2x) \quad y(0) = 1 \]

469. Solve:

\[ y'' + 6y' + 5y = 12e^x \quad y(0) = -1 \quad y'(0) = 7 \]

470. Solve:

\[ y'' - 2y' + y = 6te^t \quad y(0) = 1 \quad y'(0) = 2 \]
4.2. SOLVING INITIAL VALUE PROBLEMS

471. Solve:
\[ y'' - 2y' - 3y = 8 - 16x - 12x^2 \quad y(0) = 2 \quad y'(0) = 2 \]

472. Solve:
\[ y'' + y' - 2y = 3e^x \quad y(0) = 5 \quad y'(0) = 8 \]

473. Solve:
\[ y'' + y = e^x(3\cos(x) - \sin(x)) \quad y(0) = 3 \quad y'(0) = 5 \]

474. Solve:
\[ y'' - 2y' + y = (x + 2)e^{2x} \quad y(0) = 1 \quad y'(0) = 0 \]

475. Solve:
\[ y'' - 4y' + 5y = te^{2t} \quad y(0) = 1 \quad y'(0) = 4 \]

476. Solve:
\[ y'' - 3y' + 2y = (1 - 2x)e^x \quad y(0) = 3 \quad y'(0) = 5 \]

477. Solve:
\[ y'' - 4y' + 3y = -e^x(3\sin(x) + \cos(x)) \quad y(0) = 1 \quad y'(0) = 4 \]

478. Solve:
\[ y'' + y = 2\sin(x) + 4x\cos(x) \quad y(0) = 3 \quad y'(0) = 2 \]

479. Solve:
\[ y'' + 4y = \sin(2t) \quad y(0) = 10 \quad y'(0) = 0 \]

480. Solve:
\[ y'' - 7y' + 10y = 9\cos(x) + 7\sin(x) \quad y(0) = 5 \quad y'(0) = -4 \]

481. Solve:
\[ y'' + 4y = 4x^2 - 4x + 10 \quad y(0) = 0 \quad y'(0) = 3 \]

482. Solve:
\[ y'' + 4y' + 5y = e^{2x}\sin(x) \quad y(0) = 1 \quad y'(0) = -6 \]

483. Solve:
484. Solve:

\[ y'' - 2y' + y = 18xe^x \quad y(0) = 1 \quad y'(0) = 3 \]

485. Solve:

\[ y'' - 5y' + 6y = e^x(4\sin(x) - 2\cos(x)) \quad y(0) = 3 \quad y'(0) = 3 \]

486. Solve:

\[ y'' + y = e^{2x}(4x^3 + 12x^2 + 6x) \quad y(0) = 1 \quad y'(0) = 1 \]

487. Solve:

\[ y'' - y' - 2y = e^{2x}(x^2 + 6x + 2) \quad y(0) = 2 \quad y'(0) = 1 \]

488. Solve:

\[ y'' - 2y' + 10y = -12\cos(3x) + x\sin(x) + \sin(x) - 6x\cos(x) \quad y(0) = 1 \quad y'(0) = 13 \]

489. Solve:

\[ y''' - y'' + y' - y = -2e^t \quad y(0) = 2 \quad y'(0) = 4 \quad y''(0) = 4 \]

490. Solve:

\[ y''' - 6y'' + 11y' - 6y = e^t + e^{2t} + e^{3t} \quad y(0) = 0 \quad y'(0) = 0 \quad y''(0) = 0 \]

491. Solve:

\[ y''' + y'' + 3y' - 5y = 16e^{-x} \quad y(0) = 0 \quad y'(0) = 2 \quad y''(0) = -4 \]

492. Solve:

\[ y'' - 2y' + 4y' - 8y = 8e^{2x}\cos(2x) - 16e^{2x}\sin(2x) \quad y(0) = 4 \quad y'(0) = 8 \quad y''(0) = 0 \]

493. Solve:

\[ y''' - 8y'' + 21y' - 18y = 2e^{3x} \quad y(0) = 2 \quad y'(0) = 5 \quad y''(0) = 15 \]

494. Solve:

\[ y'' - y = \sum_{n=0}^{\infty} e^{nx} \quad y(0) = 0 \quad y'(0) = 0 \]

495. Solve:

\[ xy'' - y' = x^2 \quad y(0) = 0 \quad y(1) = 2 \]

495. Solve:

\[ xy'' + y' - 2xy = e^x \quad y(0) = 1 \quad y(1) = 2 \]
4.3. UNIT STEP FUNCTION

496. Solve:

\[ xy'' + 2xy' + 2y = 0 \quad y(0) = 0 \quad y(1) = 2 \]

497. Solve:

\[ xy'' + (x + 2)y' + y = -1 \quad y(0) = 0 \quad y(1) = 2 \]

498. Use the formula

\[ \mathcal{L}(y''(x)) = s^2 \mathcal{L}(y(x)) - sy(0) - y'(0) \]

To find the Laplace transform of

\[ y(x) = \sin(ax) \]

499. Convert the Cauchy Euler equation to the variable \( t \) and then use Laplace transforms to solve it.

\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - 4y = \sin(\ln(x)) \quad y(1) = 1 \quad y'(1) = 3 \]

4.3 Unit Step Function

So far we have learned how to use Laplace Transforms to solve differential equations involving continuous functions. We will now learn how to solve differential equations involving piecewise functions. First a definition.

The Unit Step Function is defined as:

\[ u_c(x) = \begin{cases} 0 & x < c \\ 1 & x \geq c \end{cases} \]

It can be shown that:

\[ \mathcal{L}(u_c(x)f(x)) = e^{-cs} \mathcal{L}(f(x+c)) \quad \mathcal{L}^{-1}(e^{-cs}F(s)) = u_c(x)f(x-c) \]

An Example: Solve:

\[ y'' + y = g(x) \quad y(0) = 0 \quad y'(0) = 1 \]

\[ g(x) = \begin{cases} x & x < 2 \\ 4 & 2 < x \end{cases} \]

Start by writing the equation using the unit step function:

\[ y'' + y = x + (4 - x)u_2(x) \]
Take the Laplace Transform of both sides:

\[ s^2 \mathcal{L}(y) - 1 + \mathcal{L}(y) = \frac{1}{s^2} + e^{-2s} \mathcal{L}(4 - (x + 2)) \]

\[ \mathcal{L}(y)(s^2 + 1) = \frac{1}{s^2} + e^{-2s} \left( \frac{2}{s} - \frac{1}{s^2} \right) + 1 \]

\[ \mathcal{L}(y) = \frac{1}{s^2(s^2 + 1)} + \frac{1}{(s^2 + 1)} + e^{-2s} \left( \frac{2}{s(s^2 + 1)} - \frac{1}{s^2(s^2 + 1)} \right) \]

Applying partial fraction to the first term and on the terms multiplied by \( e^{-2s} \) gives:

\[ \mathcal{L}(y) = \frac{1}{s^2} - \frac{1}{s^2 + 1} + \frac{1}{(s^2 + 1)} + e^{-2s} \left( \frac{-2s}{s^2 + 1} + \frac{1}{s^2 + 1} + \frac{2}{s} - \frac{1}{s^2} \right) \]

Calculating the inverse Laplace Transform:

\[ y = x + u_2(x) \left( -2 \cos(x) + \sin(x) + 2 - x \right) \bigg|_{x-2} \]

\[ y = x + u_2(x) \left( -2 \cos(x - 2) + \sin(x - 2) + 2 - (x - 2) \right) \]

\[ y = x + u_2(x) \left( -2 \cos(x - 2) + \sin(x - 2) - x \right) \]

500. Express the piecewise function using the unit step function and then find its Laplace Transform:

\[ f(x) = \begin{cases} 
  x & x < 4 \\
  e^{3x} & x \geq 4
\end{cases} \]

501. Express the piecewise function using the unit step function and then find its Laplace Transform:

\[ f(x) = \begin{cases} 
  \sin(2x) & x < 3 \\
  xe^{4x} & 3 \leq x < 6 \\
  \cosh(2x) & 6 \leq x
\end{cases} \]

502. Find the Inverse Laplace Transform of:

\[ F(s) = \frac{e^{-5s}}{(s + 4)} \]

503. Find the Inverse Laplace Transform of:

\[ F(s) = \frac{e^{-3s}(s - 5)}{(s + 1)(s + 2)} \]

504.

Solve:
4.3. UNIT STEP FUNCTION

\[ y' - 4y = 1 + u_x(x)x \sin(x) \quad y(0) = 1 \]

505.
Solve:

\[ y'' + 3y' + 2y = g(x) \quad y(0) = 2 \quad y'(0) = -1 \]

\[ g(x) = \begin{cases} 
  x^2 & x < 3 \\
  1 & 3 < x 
\end{cases} \]

506.
Solve:

\[ y'' + y = g(x) \quad y(0) = 2 \quad y'(0) = -1 \]

\[ g(x) = \begin{cases} 
  x & x < 1 \\
  x^2 & x > 1 
\end{cases} \]

507.
Solve:

\[ y'' + 5y' + 6y = g(x) \quad y(0) = -1 \quad y'(0) = 0 \]

\[ g(x) = \begin{cases} 
  0 & x < 1 \\
  4x & 1 < x < 5 \\
  8 & 5 < x 
\end{cases} \]

508.
Solve:

\[ y'' - 2y' + y = g(x) \quad y(0) = 1 \quad y'(0) = 3 \]

\[ g(x) = \begin{cases} 
  2x & x < 2 \\
  x^2 & 2 < x < 5 \\
  1 & 5 < x 
\end{cases} \]

509.
Solve:

\[ y'' + 8y' + 17y = g(x) \quad y(0) = 1 \quad y'(0) = 0 \]

\[ g(x) = \begin{cases} 
  x & x < 1 \\
  x^2 & 1 < x < 5 \\
  x & 5 < x 
\end{cases} \]

510.
Solve:

\[ y'' + 4y = g(x) \quad y(0) = 0 \quad y'(0) = 0 \]

\[ g(x) = \begin{cases} 
  x & x < 1 \\
  x^2 & 1 < x < 2 \\
  1 & 2 < x 
\end{cases} \]

511.
Solve:

\[ y'' + y = g(x) \quad y(0) = 0 \quad y'(0) = 1 \]

\[ g(x) = \begin{cases} 
  \cos(2x) & x < \frac{\pi}{2} \\
  \sin(2x) & \frac{\pi}{2} < x 
\end{cases} \]

512.
Solve:

\[ y'' + 4y = g(x) \quad y(0) = -3 \quad y'(0) = 1 \]

\[ g(x) = \begin{cases} 
  |\sin(x)| & x < 2\pi \\
  0 & x > 2\pi 
\end{cases} \]

513.
Show: if

\[ Y(s) = \sum_{n=0}^{\infty} e^{-ns} \left( \frac{1}{s^2} + \frac{n}{s} \right) \]

Then

\[ \mathcal{L}^{-1}\left( Y \right) = \begin{cases} 
  t & x < 1 \\
  2t & 1 < x < 2 \\
  3t & 2 < x < 3 \\
  \vdots \\
  nt & n - 1 < x < n 
\end{cases} \]

514.
Express the floor function as a sum of unit step functions and find its Laplace Transform.

### 4.4 Convolution

If you are asked to find the inverse Laplace Transform of \( F(s)G(s) \) and you know the inverse Laplace Transform of both \( F(s) \) and \( G(s) \) what is the relationship between the Laplace Transform of \( F(x)G(s) \) and Laplace Transform of \( F(s) \) and \( G(s) \)? The answer is a new operation called Convolution.
The Convolution of \( f(x) \) and \( g(x) \) is denoted \( f * g \) and is given by the integral:

\[
f * g = \int_0^x f(x-v)g(v)\,dv
\]

For example the Convolution of \( x^2 \) and \( x^3 \) is:

\[
x^2 * x^3 = \int_0^x (x-v)^2 v^3\,dv = \int_0^x (x^2 - 2xv + v^2)v^3\,dv = \int_0^x (x^2v^4 - 2xv^5 + v^6)\,dv = \left[ \frac{x^6}{6} \right]_0^x = \frac{x^6}{60}
\]

Some basic properties of Convolution are:

\[
f * g = g * f \quad f * (g + h) = f * g + f * h \quad (f * g) * h = f * (g * h) \quad f * 0 = 0
\]

It is also true that

\[
f * 1 \neq f
\]

since

\[
f * 1 = \int_0^x f(x-v)\,dv \neq f
\]

What makes Convolution useful in differential equations are the following properties involving the Laplace Transform:

\[
\mathcal{L}(f * g) = \mathcal{L}(f) \cdot \mathcal{L}(g) \quad (\mathcal{L}^{-1}(F(s)G(s))) = ((\mathcal{L}^{-1}(F(s))) * ((\mathcal{L}^{-1}(G(s)))
\]

To show the first property:

\[
\mathcal{L}(f * g) = \mathcal{L}(f) \cdot \mathcal{L}(g)
\]

holds we let

\[
F(s) = \mathcal{L}(f) = \int_0^\infty f(x)e^{-sx}\,dx \quad G(s) = \mathcal{L}(f) = \int_0^\infty g(y)e^{-sy}\,dy
\]

\[
\mathcal{L}(f * g) = \int_0^\infty e^{-st} \left( \int_0^t f(t-v)g(v)\,dv \right)\,dt = \int_0^\infty e^{-st} \left( \int_0^\infty u_{t-v}(t)f(t-v)g(v)\,dv \right)\,dt
\]

Remember \( u_{t-v}(t) \) will be zero if \( v > t \)

Reversing the order if integration gives

\[
\mathcal{L}(f * g) = \int_0^\infty g(v) \left( \int_0^\infty e^{-st} u_{t-v}(t)f(t-v)\,dt \right)\,dv
\]

The integral in parentheses is equal to \( e^{-sv}F(s) \) so we have:

\[
\mathcal{L}(f * g) = \int_0^\infty g(v)e^{-sv}F(s)\,dv = F(s)\int_0^\infty g(v)e^{-sv}\,dv
\]
So
\[ \mathcal{L}(f * g) = F(s) \cdot G(s) \]

An Example: Solve:
\[ y' - 2 \int_0^x e^{x-v} y(v) dv = x \quad y(0) = 2 \]

Rewriting the integral as convolution
\[ y' - 2e^x * y = x \quad y(0) = 2 \]

Taking the Laplace Transform of each side gives:
\[ s\mathcal{L}(y) - 2 - 2\left( \frac{1}{s-1}\mathcal{L}(y) \right) = \frac{1}{s^2} \]

Solving for \( \mathcal{L}(y) \)
\[ (s - 1)s\mathcal{L}(y) - 2\mathcal{L}(y) = \frac{s - 1}{s^2} + 2(s - 1) \]
\[ \mathcal{L}(y)(s^2 - s - 2) = \frac{s - 1}{s^2} + 2(s - 1) \]
\[ \mathcal{L}(y) = \frac{s - 1}{s^2(s^2 - s - 2)} + \frac{2(s - 1)}{s^2 - s - 2} \]
\[ \mathcal{L}(y) = \frac{s - 1}{s^2(s - 2)(s + 1)} + \frac{2(s - 1)}{(s - 2)(s + 1)} \]

After partial fractions we have:
\[ \mathcal{L}(y) = \frac{2}{s + 1} + \frac{3}{4} \frac{1}{s - 2} - \frac{3}{4} \frac{1}{s} + \frac{1}{2} \frac{1}{s^2} \]

Taking the inverse Laplace Transform
\[ y = 2e^{-x} + \frac{3}{4}e^{2x} - \frac{3}{4} + \frac{1}{2}x^2 \]

Convolution can also be used to find the output response \( y(x) \) of a physical system for some input function \( f(x) \). Consider the physical system governing the motion of a mass connected to a spring:
\[ my'' + by' + ky = f(x) \]

For Simplicity lets assume
\[ y(0) = y'(0) = 0 \]
4.4. CONVOLUTION

Taking the Laplace transform of both sides of the differential equation gives

\[ ms^2 \mathcal{L}(y) + bs\mathcal{L}(y) + k\mathcal{L}(y) = \mathcal{L}(f(x)) \]

\[ \mathcal{L}(y) = \frac{\mathcal{L}(f(x))}{ms^2 + bs + k} \]

The function

\[ W(s) = \frac{1}{ms^2 + bs + k} \]

is called the Transfer Function for the physical system.

\[ \mathcal{L}(y) = \mathcal{L}(f(x))W(s) \]

The function

\[ w(x) = \mathcal{L}^{-1}\left(W(s)\right) \]

is called the Weight Function of the system.

Through convolution we see the solution to the differential equations is

\[ y(x) = \int_{0}^{x} w(u)f(x - u)du \]

This equation reduces solving the same mass spring system for different input functions into solving a definite integral for each input function.

515. Find:

\[ x * e^x \]

516. Find:

\[ \sin(x) * \sin(2x) \]

517. If \( y(0) = 0 \) show:

\[ 2 * (y(x) \cdot y'(x)) = y^2(x) \]

518. Find the Laplace Transform of:

\[ f(x) = \int_{0}^{x} (x - v)^2 e^{5v} dv \]

519. Find the Laplace Transform of:
\[ f(x) = \int_0^x \cos(x - v) \cdot \sin(4v) \, dv \]

520.
Find the inverse Laplace Transform of the following function by using Convolution:

\[ F(s) = \frac{1}{s^2(s^2 + 1)} \]

521.
Find the inverse Laplace Transform of the following function by using Convolution:

\[ F(s) = \frac{s}{(s^2 + 1)^2} \]

522.
Solve:

\[ y' + \int_0^x (x-v)y(v) \, dv = t \quad y(0) = 0 \]

523.
Solve:

\[ y' + y - \int_0^x \sin(x - v)y(v) \, dv = -\sin(x) \quad y(0) = 1 \]

524.
Solve:

\[ y' = 1 - \sin(x) \int_0^x y(v) \, dv \quad y(0) = 0 \]

525.
Solve:

\[ y'' + y = 2x \cdot e^x \quad y(0) = y'(0) = 0 \]

526.
Solve:

\[ y'' - y = 2x \cdot \sin(x) \quad y(0) = 4 \quad y'(0) = 2 \]

527.
Solve using convolution. Leave your answer in terms of an integral involving \( g(x) \).

\[ y'' + y = g(x) \quad y(0) = 1 \quad y'(0) = 1 \]

528.
Use convolution to evaluate the following integral

\[ \int_0^1 (t-u)^5 u^8 \, du \]
4.4. CONVOLUTION

529.
Use convolution to evaluate the following integral

\[ \int_0^t (t-u)^n u^m \, du \]

530.
If \( f \) and \( g \) have the following properties

\[ f(x) * f(x) = xf(x) \quad f(0) = 4 \]

Find \( f(x) \)

531.
Without using the definition of convolution calculate

\[ f(t) = u_1(t) * u_2(t)t^2 \]

By calculating the Laplace Transform of \( f(t) \) simplifying and then calculating the inverse Laplace transform to find \( f(t) \)

532.
If

\[ f(x) * f'(x) = \frac{x^2}{2} \quad f(0) = f'(0) = 1 \]

Find \( f(x) \)

533.
Prove The following with mathematical induction

\[ \underbrace{x * x * x * \ldots * x}^\text{n-times} = x^{2n-1} \frac{1}{(2n-1)!} \]

534.
Show the following property holds

\[ t * t^n = \frac{t^{n+2}}{(n+1)(n+2)} \]

535.
If \( f(x) = (1 + x)^2 \)

\[ f(x) * g(x) = f(x) \cdot g(x) \quad g(0) = e^{-1} \]

Find \( g(x) \)

536.
Show

\[ f * g' = f' * g \]
4.5 Delta Function

The Dirac Delta Function $\delta(x)$ is defined as

$$
\delta(x) = \begin{cases} 
0 & x \neq 0 \\
\infty & x = 0 
\end{cases}
$$

And has the property:

$$
\int_{-\infty}^{\infty} f(x) \delta(x) dx = f(0)
$$

$$
\mathcal{L}(\delta(x - c)) = e^{-cs}
$$

The delta function shows up in science when considering the impulse of a force over a short interval. If a force $F(t)$ on the time interval $t_0$ to $t_1$ then the impulse due to the force is:

$$
\text{Impulse} = \int_{t_0}^{t_1} F(t) dt
$$

By Newton’s second law

$$
\int_{t_0}^{t_1} F(t) dt = \int_{t_0}^{t_1} m \frac{dv}{dt} dt = mv(t_1) - mv(t_0)
$$

where $m$ is the mass and $v$ is the velocity. Since an object’s momentum is the product of mass and velocity we see that the impulse is equal to the change in momentum.

An Example: Solve:

$$
y'' + y = \delta(x - \pi) \quad y(0) = 0 \quad y'(0) = 0
$$

Taking the Laplace Transform of both sides gives:

$$
\mathcal{L}(y'' + y) = \mathcal{L}(\delta(x - \pi))
$$

$$
\mathcal{L}(y) (s^2 + 1) = e^{-\pi s}
$$

Finding the inverse Laplace Transform:

$$
y = u_\pi(x) \sin(x) \bigg|_{x = \pi} = u_\pi(x) \sin(x - \pi)
$$

After a trig identity

$$
y = -u_\pi(x) \sin(x)
$$
4.5. DELTA FUNCTION

537. Find:

\[ \int_0^3 e^{x^2} \delta(x - 1) \, dx \]

538. Find the value of \( k \) so that

\[ \int_0^1 \sin^2(\pi(x - k)) \delta\left( x - \frac{1}{2} \right) \, dx = \frac{3}{4} \]

539. Find

\[ \mathcal{L}\{\delta(\sin(\pi x))\} \]

540. Solve

\[ y' + y = \delta(x - 1) \quad y(0) = 2 \]

541. Solve

\[ y'' - 4y' + 4y = e^{2x} + \delta(x - 2) \quad y(0) = 1 \quad y'(0) = 3 \]

542. Solve

\[ y'' + 2y' + 2y = \delta(x - \pi) \quad y(0) = 1 \quad y'(0) = 1 \]

543. Solve

\[ y'' + 2y' - 3y = \delta(x - 1) - \delta(x - 2) \quad y(0) = 2 \quad y'(0) = -2 \]

544. Solve

\[ y'' - y' - 2y = 3\delta(x - 1) + e^x \quad y(0) = 0 \quad y'(0) = 3 \]

545. Solve

\[ y'' - 3y' + 2y = \delta(x - 1) - \delta(x - 2) \quad y(0) = 2 \quad y'(0) = 3 \]

546. Solve

\[ y'' + y = \delta\left( x - \frac{\pi}{2} \right) - \delta\left( x - \frac{3\pi}{2} \right) \quad y(0) = 0 \quad y'(0) = 0 \]

547. Solve

\[ y'' - 4y' + 13y = xu_2(x) + \delta\left( x - 2 \right) \quad y(0) = 1 \quad y'(0) = 1 \]

548. Solve

\[ y' - y = x \ast e^x + \delta\left( x - 1 \right) \quad y(0) = 1 \]

549. Solve

\[ y'' + y = \sum_{n=0}^{\infty} \delta\left( x - n\pi \right) \quad y(0) = 0 \quad y'(0) = 0 \]

550. Show

\[ \mathcal{L}\{\delta(x - n) \ast \delta(x - m)\} = \mathcal{L}\{\delta(x - m - n)\} \]
In this chapter we will study systems of differential equation of the form:

\[
\frac{dx}{dt} = F(t, x, y) \quad \frac{dy}{dt} = G(t, x, y)
\]

Where the dependent variables \(x\) and \(y\) are linked together by the independent variable \(t\).

5.1 Linear Systems

The theory of first order linear systems is very similar to the theory of second order equation that we studied in chapter 2. Consider the following system of equations:

\[
\frac{dx}{dt} = 4x - y; \quad \frac{dy}{dt} = 2x + y
\]

This system can be transformed into a second order equation by solving for \(y\) in the first equation and substituting it into the second:

\[
y = 4x - \frac{dx}{dt} \quad \frac{d}{dt} \left(4x - \frac{dx}{dt}\right) = 2x + 4x - \frac{dx}{dt}
\]

This produces the second order equation:

\[
\frac{4}{dt} \frac{dx}{dt} - 5 \frac{dx}{dt^2} = 2x + 4x - \frac{dx}{dt}
\]

Which simplifies to

\[
\frac{d^2x}{dt^2} - 5 \frac{dx}{dt} + 6x = 0
\]

This has the characteristic equation:
\[ r^2 - 5r + 6 = 0 \quad (r - 3)(r - 2) = 0 \]

Giving the solution:

\[ x_1 = e^{2t} \quad x_2 = e^{3t} \]

Substituting these equations into \( y = 4x - \frac{dx}{dt} \) gives two solutions for \( y \)

\[ y_1 = 2e^{2t} \quad y_2 = e^{3t} \]

Making the general solution:

\[ x = C_1e^{3t} + C_2e^{2t} \quad y = C_1e^{3t} + 2C_2e^{2t} \]

In chapter 2 we learned that if the Wronskian of the two solutions to a second order equation is nonzero on an interval then the two solutions are linearly independent and form a Fundamental Solution Set. For systems of equations the Wronskian is:

\[ W(t) = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_1y_2 - x_2y_1 \]

It can be shown that if the homogenous system:

\[ \frac{dx}{dt} = a_1(t)x(t) + b_1y(t) \quad \frac{dy}{dt} = a_2(t)x(t) + b_2y(t) \]

has solutions:

\[ x = x_1(t) \quad y = y_1(t) \quad \text{and} \quad x = x_2(t) \quad y = y_2(t) \]

and the Wronskian is non zero on the interval \([a, b]\) then:

\[ x = x_1(t) \quad y = y_1(t) \quad \text{and} \quad x = x_2(t) \quad y = y_2(t) \]

is the general solution to the system of differential equations on the interval \([a, b]\). We see that the Wronskian for the last problem we solved is:

\[ W(t) = \begin{vmatrix} e^{3t} & e^{2t} \\ e^{3t} & 2e^{2t} \end{vmatrix} = e^{5t} \neq 0 \]

So the general solution is indeed:

\[ x = C_1e^{3t} + C_2e^{2t} \quad y = C_1e^{3t} + 2C_2e^{2t} \]

You can also solve the same system of differential equations using matrices. Let us first write the system of equations as a matrix equation of the form:

\[ \mathbf{Ax} = \mathbf{x}' \quad \text{where} \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \]
\[
\frac{dx}{dt} = 4x - y; \quad \frac{dy}{dt} = 2x + y
\]

The system in matrix form is:

\[
\begin{bmatrix}
4 & -1 \\
2 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
x' \\
y'
\end{bmatrix}
\]

As we earlier showed the system of equations can be reduced to a second order equation with constant coefficients whose solutions were of the form: \(e^{rt}\). It is reasonable to assume the above matrix equation will have a solution of the form:

\[
x(t) =
\begin{bmatrix}
x \\
y
\end{bmatrix} = e^{rt}u
\]

Where \(r\) is a constant and \(u\) is a constant nonzero vector. Substituting \(x(t) = e^{rt}u\) into \(Ax = x'\) gives:

\[
re^{rt}u = Ae^{rt}u
\]

Dividing by the nonzero factor \(e^{rt}\) and rearranging the terms gives:

\[
(A - rI)u = 0
\]

The values of \(r\) and \(u\) that satisfy the above equation are the eigenvalues and eigenvectors of the matrix \(A\). To find the eigenvalues \(r\) of a matrix we take the determinant of both sides to the above equation:

\[
\begin{vmatrix}
(A - rI)
\end{vmatrix} = |0| = 0
\]

Since \(u\) is nonzero the solutions: \(r\), to the above equation come from the solutions to:

\[
\begin{vmatrix}
(A - rI)
\end{vmatrix} = 0
\]

This equation is a polynomial with the variable \(r\). This equation is called the Characteristic Equation whose roots are the eigenvalues of the matrix \(A\). Back to the problem we were solving. Our characteristic equation is:

\[
\begin{vmatrix}
4 - r & -1 \\
2 & 1 - r
\end{vmatrix} = (4 - r)(1 - r) + 2 = 0
\]

Factoring the characteristic equation and solving gives:

\[
(r - 2)(r - 3) = 0 \quad r_1 = 2 \quad r_2 = 3
\]

NOTE: this is the same characteristic equation we got in our first solution to this problem.

Now that we have the eigenvalues for \(A\) we need the eigenvectors \(u\). To find the eigenvector for each eigenvalue we must solve the following equation for \(u\):

\[
(A - rI)u = 0
\]

For \(r_1 = 2\) we have:
\[
\begin{pmatrix}
2 & -1 \\
2 & -1 \\
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
\end{pmatrix}
= 0
\]

We see the solution to this matrix equation is \( y = 2x \). Taking \( x \) to be 1 we get our eigenvector:

\[
u_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\]

For \( r_2 = 3 \) we have:

\[
\begin{pmatrix}
1 & -1 \\
2 & -2 \\
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
\end{pmatrix}
= 0
\]

We see the solution to this matrix equation is \( y = x \). Taking \( x \) to be 1 we get our eigenvector:

\[
u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

Making the solution to the differential equation:

\[
\begin{pmatrix}
x \\
y \\
\end{pmatrix}
= C_1 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

Making:

\[x = C_1 e^{3t} + C_2 e^{2t} \quad y = C_1 e^{3t} + 2C_2 e^{2t}\]

The same solution we obtained earlier.

The matrix containing the two eigenvectors as its two columns is called the Fundamental Matrix. The fundamental matrix for our problem is:

\[
X(t) = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}
\]

Some times the matrix has complex eigenvalues: \( r = \alpha \pm \beta i \) and eigenvectors \( a \pm bi \), the solution becomes:

\[x(t) = C_1 \left( e^{\alpha t} \cos(\beta t)a - e^{\alpha t} \sin(\beta t)b \right) + C_2 \left( e^{\alpha t} \cos(\beta t)b + e^{\alpha t} \sin(\beta t)a \right)\]

Example:

Solve the system of differential equations:

\[x' = -x - 2y \quad y' = 8x - y\]

Writing the system as a matrix equation gives:

\[
\begin{pmatrix}
-1 & -2 \\
8 & -1 \\
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
\end{pmatrix}
= \begin{pmatrix} x' \\ y' \end{pmatrix}
\]

Find the characteristic equation:
Which has roots:

\[ r = -1 \pm 4i \]

Using \( r = -1 + 4i \) the eigenvector is:

\[
\begin{bmatrix}
-4i & -2 \\
8 & -4i
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} = 0
\]

This has the solution:

\[ y = -2x \quad \text{taking } x = 1 \text{ the eigenvector is} \]

\[
\mathbf{u}_1 = \begin{bmatrix} 1 \\ -2i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -2 \end{bmatrix}
\]

We could find the eigenvector corresponding to \( r = -1 - 4i \) in the same way we found the first eigenvector but we do not need to. It is true that if \( \mathbf{u}_1 = \mathbf{a} + i\mathbf{b} \) is an eigenvector for eigenvalue \( r_1 = \alpha + \beta i \) then \( \mathbf{u}_2 = \mathbf{a} - i\mathbf{b} \) is an eigenvector for \( r_1 = \alpha - \beta i \). This means the eigenvector for \( r = -1 - 4i \) is

\[
\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 2 \end{bmatrix}
\]

Making the solution:

\[
\mathbf{x}(t) = C_1 \left( e^{-t} \cos(4t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} - e^{-t} \sin(4t) \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right) + C_2 \left( e^{-t} \cos(4t) \begin{bmatrix} 0 \\ 2 \end{bmatrix} + e^{-t} \sin(4t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)
\]

551. Solve:

\[ x' = 2x + y \quad y' = -x + 4y \]

552. Solve:

\[ x' = -6x + 4y \quad y' = -5x + 3y \]

553. Solve:

\[ x' = 2x + 4y \quad y' = -5x - 2y \]

554. Solve:
555.
Solve:
\[ x' = -x - 2y \quad y' = 13x + y \]

556.
Solve:
\[ x' = -7x - 2y \quad y' = 5x - y \]

557.
Solve:
\[ x' = x + 3y \quad y' = 12x + y \]

558.
Solve:
\[ x' = x + 2y + 2z \quad y' = 2x + 3y \quad z' = 2x + 3y \]

559.
Solve:
\[ x' = 2x - 4y \quad y' = 2x - 2y \]

560.
Solve:
\[ x' = -2x - 5y \quad y' = x + 2y \]

561.
Solve:
\[ x' = 4x - y \quad y' = x + 2y \]

562.
Solve:
\[ x' = -x + 5y \quad y' = -x + 3y \]

Convert the following differential equation governing the motion of a mass attached to a spring into a system of equations:

\[ my'' + by' + ky = 0 \]

563.
Solve:
\[
\begin{bmatrix}
2 & -3 \\
1 & -2 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
\end{bmatrix}
=
\begin{bmatrix}
x' \\
y' \\
\end{bmatrix}
\]
5.2. LOCALLY LINEAR EQUATIONS

564. Solve:

\[
\begin{bmatrix}
-1 & 2 \\
-1 & -3
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
x' \\
y'
\end{bmatrix}
\]

565. Use the definition of the Wronskian in this section to prove Abel’s Theorem for systems: If \(\{y_1, y_2\}\) are solutions to

\[x' = Ax\]

Then

\[W(y_1, y_2) = Ce^{\int_{t_0}^{t} \text{tr}(A) \, du}\]

5.2 Locally Linear Equations

We will now discuss nonlinear systems around their critical or equilibrium points. The general system of equations we will be studying are systems of the form:

\[
x' = F(x, y) \quad y' = G(x, y)
\]

The Critical or Equilibrium Points of this system are the values \((x_0, y_0)\) such that

\[x' = F(x_0, y_0) = 0 \quad \text{and} \quad y' = G(x_0, y_0) = 0\]

If both \(x_0 > 0\) and \(y_0 > 0\) then the critical point is defined to be a Positive Equilibrium Point and will be discussed when we consider the competing species problem.

The system is said to be Locally Linear around a critical or equilibrium point if both \(F\) and \(G\) have continuous first and second order partial derivatives at the critical point \((x_0, y_0)\).

For the general system:

\[
x' = F(x, y) \quad y' = G(x, y)
\]

with equilibrium point \((x_0, y_0)\) we can form the locally linear system in matrix form:

\[
\begin{bmatrix}
F_x(x_0, y_0) & F_y(x_0, y_0) \\
G_x(x_0, y_0) & G_y(x_0, y_0)
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
x' \\
y'
\end{bmatrix}
\end{bmatrix}
\]

With the matrix:

\[
J = \begin{bmatrix}
F_x(x_0, y_0) & F_y(x_0, y_0) \\
G_x(x_0, y_0) & G_y(x_0, y_0)
\end{bmatrix}
\]

called the Jacobian Matrix.
**An Example**

Find the positive equilibrium solution to the system and discuss the stability near this point.

\[
\frac{dx}{dt} = x \left(4 - 2y\right) \quad \frac{dy}{dt} = y \left(-2 + x\right)
\]

We see that the positive equilibrium solutions come from the equations:

\[
4 - 2y = 0 \quad -2 + x = 0
\]

Giving the positive equilibrium solution \((2, 2)\). The Jacobian Matrix for this problem is:

\[
J = \begin{bmatrix} 4 - 2y & -2x \\ y & x - 2y \end{bmatrix}
\]

The Jacobian Matrix at the equilibrium point \((2, 2)\) is:

\[
J = \begin{bmatrix} 0 & -4 \\ 2 & -2 \end{bmatrix}
\]

Producing the locally linear system:

\[
\begin{bmatrix} 0 & -4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}
\]

Now we need the eigenvalues and eigenvector of our Jacobian Matrix evaluated at the equilibrium points. The characteristic equation is:

\[
\det(J - rI) = -r(-2 - r) + 8 = 0 \quad r^2 + 2r + 8 = 0
\]

Which has roots:

\[
r_{1,2} = -1 \pm \sqrt{7}i
\]

I have no desire to find the eigenvectors so, instead of the two eigenvectors I present you with one rabbit with two pancakes on its head.
Find all equilibrium solutions and discuss the stability of each point and interpret the results of the competing species model.

\[
\frac{dx}{dt} = x \left( 1 - x - y \right) \quad \frac{dy}{dt} = y \left( 2 - y - 3x \right)
\]

Find all equilibrium solutions and discuss the stability of each point and interpret the results of the competing species model.

\[
\frac{dx}{dt} = x \left( 2 - x - y \right) \quad \frac{dy}{dt} = y \left( 3 - 2x - 4y \right)
\]

Find all equilibrium solutions and discuss the stability of each point and interpret the results of the competing species model.

\[
\frac{dx}{dt} = x \left( 2 - 2x - y \right) \quad \frac{dy}{dt} = y \left( 3 - 2x - 2y \right)
\]
570. Find all equilibrium solutions and discuss the stability of each point and interpret the results of the competing species model

\[
\frac{dx}{dt} = x(3 - x - y) \quad \frac{dy}{dt} = y(4 - x - 2y)
\]

571. Find all equilibrium solutions and discuss the stability of each point and interpret the results of the competing species model

\[
\frac{dx}{dt} = x\left(2 - x - \frac{1}{2}y\right) \quad \frac{dy}{dt} = y\left(3 - x - y\right)
\]

572. Find all equilibrium solutions and discuss the stability of each point and interpret the results of the competing species model

\[
x' = x(2 - x - y) \quad y' = y(3 - 2y - x)
\]

573. Find all equilibrium solutions and discuss the stability of each point and interpret the results of the competing species model

\[
x' = x(2 - x - 2y) \quad y' = y(7 - 2y - 6x)
\]

574. Find all equilibrium solutions and discuss the stability of each point and interpret the results of the competing species model

\[
x' = x(4 - 2x - 2y) \quad y' = y(5 - y - 4x)
\]

575. Find all equilibrium solutions and discuss the stability of each point and interpret the results of the competing species model

\[
x' = x(4 - x - 2y) \quad y' = y(3 - y - x)
\]

576. Find the positive equilibrium solution to the system and discuss the stability near this point.

\[
\frac{dx}{dt} = x\left(1 - x - y\right) \quad \frac{dy}{dt} = y\left(3 - x - 2y\right)
\]

577. Find the positive equilibrium solution to the competing species system of equations and discuss the stability near this point.

\[
\frac{dx}{dt} = x\left(2 - \frac{x}{3} - \frac{4y}{x + 4}\right) \quad \frac{dy}{dt} = y\left(\frac{x}{x + 4} - \frac{1}{2}\right)
\]
Find the equilibrium solution to the system and sketch phase plane diagram.

\[
\frac{dx}{dy} = -y(y - 2) \quad \frac{dy}{dt} = (x - 2)(y - 2)
\]

Solve the system of equations by converting it into a first order differential equation

\[
\frac{dx}{dy} = xy - \frac{x^3}{y} \quad \frac{dy}{dt} = 3x^2 - y^2
\]

Solve the system of equations by converting it into a first order differential equation

\[
\frac{dx}{dt} = x \quad \frac{dy}{dt} = 2y - x^3y^2
\]

5.3 Linear Systems and the Laplace Transform

We can also use Laplace Transforms to solve linear systems with initial conditions. The procedure will be quite similar to the procedure used to solve initial value problems in chapter 4. Given a system of equations we will start by taking the Laplace Transform of each side of the equation, solving for the Laplace Transform of both \(x(t)\) and \(y(t)\) and then using a table to find the inverse.

**An Example:** Solve:

\[
\frac{dx}{dt} = 2y + 4t \quad x(0) = 4 \quad \frac{dy}{dt} = 4x - 2y - 4t - 2 \quad y(0) = -5
\]

Taking the Laplace Transform of both sides of the differential equation involving \(x'\) gives:

\[
\mathcal{L}(x') = \mathcal{L}(y) + \mathcal{L}(4t)
\]

Using a table we get:

\[
s\mathcal{L}(x) - x(0) = 2\mathcal{L}(y) + \frac{4}{s^2}
\]

\[
s\mathcal{L}(x) - 4 = 2\mathcal{L}(y) + \frac{4}{s^2}
\]

Taking the Laplace Transform of both sides of the differential equation involving \(y'\) gives:

\[
\mathcal{L}(y') = 4\mathcal{L}(x) - 2\mathcal{L}(y) - \mathcal{L}(4t) - \mathcal{L}(2)
\]

\[
s\mathcal{L}(y) - y(0) = 4\mathcal{L}(x) - 2\mathcal{L}(y) - \frac{4}{s^2} - \frac{2}{s}
\]

\[
s\mathcal{L}(y) + 5 = 4\mathcal{L}(x) - 2\mathcal{L}(y) - \frac{4}{s^2} - \frac{2}{s}
\]
We now have two equations with two desired variables: $L(x)$ and $L(y)$ that we will need to solve for. In our first equation I will solve for $L(x)$ and insert it into the second.


g \frac{d}{ds} L(x) - 4 = 2L(y) + \frac{4}{s^2}

g \frac{d}{ds} L(x) = \frac{2}{s} L(y) + \frac{4}{s^3} + \frac{4}{s}

Inserting $L(x)$ into the second of our two equations gives:


g \frac{d}{ds} L(y) + 5 = 4 \frac{d}{ds} L(x) - \frac{4}{s^2} - \frac{2}{s}

g \frac{d}{ds} L(y) + 5 = 4 \left( \frac{2}{s} L(y) + \frac{4}{s^3} + \frac{4}{s} \right) - \frac{4}{s^2} - \frac{2}{s}

Multiplying both sides by $s^3$ and simplifying the result gives:

\[ L(y) \left( s^4 + 2s^3 - 8s^2 \right) = 16 - 4s^2 + 14s^2 - 5s^3 \]

\[ L(y) = \frac{16 - 4s^2 + 14s^2 - 5s^3}{s^4 + 2s^3 - 8s^2} \]

Applying partial fractions to the right hand side gives:

\[ L(y) = -\frac{6}{(x+4)} + \frac{1}{(s-2)} - \frac{2}{s} \]

Using a table to find the inverse Laplace gives:

\[ y = -6e^{-4t} + e^{2t} - 2t \]

Now that we have $y$ we can substitute its equation into the original equation for $x'$:

\[ \frac{dx}{dt} = 2y + 4t \]

becomes:

\[ \frac{dx}{dt} = 2(-6e^{-4t} + e^{2t} - 2t) + 4t \]

\[ \frac{dx}{dt} = -12e^{-4t} + 2e^{2t} \]

Integrating gives:

\[ x = 3e^{-4t} + e^{2t} + C \]

Applying the initial condition shows $C = 0$ and our solutions are:

\[ x = 3e^{-4t} + e^{2t} \quad y = -6e^{-4t} + e^{2t} - 2t \]
5.3. LINEAR SYSTEMS AND THE LAPLACE TRANSFORM

582. Solve:
\[ x' = x + y \quad x(0) = 0 \quad y' = x + y - te^t \quad y(0) = 1 \]

583. Solve:
\[ x' = y - x + 4t^2 + 1 \quad x(0) = 1 \quad y' = x - y + 2t^2 + 2t - 1 \quad y(0) = 0 \]

584. Solve:
\[ x' = 2x + y - t - 1 \quad x(0) = 1 \quad y' = y - t \quad y(0) = 1 \]

585. Solve:
\[ x' = x + y - t \sin(t) + t \cos(t) \quad x(0) = 0 \quad y' = x - t \sin(t) + \cos(t) \quad y(0) = 0 \]

586. Solve:
\[ x' = 2x - y + 1 - 2t \quad x(0) = 1 \quad y' = 2y - x + t \quad y(0) = 1 \]

587. Solve:
\[ x' = x - y - t^3 + 4t^2 + 1 \quad x(0) = 0 \quad y' = 2x - y - 2t^3 + t^2 + t + 1 \quad y(0) = 0 \]

588. Solve:
\[ x' = x + y + 2te^t \quad x(0) = 0 \quad y' = x' - t^2 e^t - 3te^t \quad y(0) = 1 \]

589. Solve:
\[ x' = x + y \quad x(0) = 0 \quad y' = x' - x \quad y(0) = 1 \]

590. Solve:
\[ x' = y - 3t \quad x(0) = 0 \quad y' = x' + 6x - 6t^3 - 3t^2 + 2 \quad y(0) = 1 \]