Linear Programming Formulation[1]

1 Mathematical Models

Model: A structure which has been built purposefully to exhibit features and characteristics of some other object such as a "DNA model" in biology, a "building model" in civil engineering, a "play in a theatre" and a "mathematical model" in operations management (research).

Why to build models?

1. Improved understanding and communication
2. Experimentation
3. Standardization for analysis

Example 1. *The formula $F = m \cdot a$ is a physical model studying the relationship between force (F), mass (m) and acceleration (a). Note that the model does not capture the friction. We say "friction" is abstracted out in this model for "computability".*

Example 2. *The 1968 movie "Planet of Apes" is a model depicting how the life on earth might turn out to be after a nuclear war among humen. According to the movie, earth in the future is controlled by apes who develop mental skills through evolution. The movie does not explain how this evolution started or proceeded, so we can say that "mental evolution of apes" is abstracted out in the 1968 movie. A follow up movie in 2011 called "Rise of the Planet of Apes" is another model explaining how the ape evolution might actually start with human intervention. It is customary to make several models of the same phonemenon to focus on different aspects such as post-nuclear-war life on earth and start of the ape evolution.*

We use variables and equations to construct mathematical models. Thus, Example 1 illustrates a mathematical model. Example 2 discusses movies as models of real life.

Common features of mathematical models:

- Abstraction, are details overlooked?
- Computability, can the model be manipulated with ease?
- Inputs, data requirements
- Uncertainty, are inputs and relationships between them uncertain?
- Decision horizons, flexibility, risk considerations ...

In deterministic mathematical models, there is no uncertainty. Then, the important concerns are abstraction and computability. More abstraction yields more computability: overlook at details to make necessary computations. But more abstraction also reduces the fidelity of the model to the real-life context.

A modeler must strike a balance between abstraction and fidelity; this is where modelling becomes more of an art form than a scientific process. This also makes it difficult to provide a recipe for modelling a real-life context. This difficulty does not really become apparent in a course where students are given a

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description of the real-life context. In a real-life project such description does not exist and its modeler's duty to construct a description by involving all the parties involved. Generally, finding a good description requires an iterative process from one description to another. During this process, the modeler adds or removes model components to balance fidelity against abstraction.

How much fidelity can there be in your models?

- Too little. Claim: some data (especially utilities, social values or feelings) can not be quantified. For example, how would you quantify the shame of selling a faulty product. Response: model what you can quantify.

- Too little. Claim: uncertain inputs. For example, we do not know gasoline prices in the future so we cannot transportation models. Response: uncertainty of inputs can be camptured by mathematical models that are not dicussed in this note.

- Too much. Claim: Models are divine, implement the outputs without hesitation. Response: Question the assumptions and the output. Regard the output as a reasonable option. Output is as good as your assumptions, if not worse. Carefully construct a model in the beginning and fearlessly revise it to improve.

It will be clear in the remainder that models combined with right data can be amazingly helpful to decision makers. But before proceeding, we can pause to reflect on what models cannot do. On this point, D. Brooks (NY Times, Feb 18 2013) wrote an op-ed article titled "What Data Can't Do". There Brooks starts by saying that data fail to capture trust – which, according to him, "is reciprocity coated by emotion". Then he categorizes situations where data have challenges. Following situations are quoted from his article:

- *Data struggles with the social.* Your brain is pretty bad at math [tedious algebra] (quick, what's the square root of 437), but it's excellent at social cognition. People are really good at mirroring each others emotional states, at detecting uncooperative behavior and at assigning value to things through emotion. Computer-driven data analysis, on the other hand, excels at measuring the quantity of social interactions but not the quality. Network scientists can map your interactions with the six co-workers you see during 76 percent of your days, but they can't capture your devotion to the childhood friends you see twice a year, let alone Dantes love for Beatrice [Portinari 1266-90, inspiration for Divine Comedy], whom he met twice.

- *Data struggles with context.* Human decisions are not discrete events. They are embedded in sequences and contexts. The human brain has evolved to account for this reality. People are really good at telling stories that weave together multiple causes and multiple contexts. Data analysis is pretty bad at narrative and emergent thinking, and it cannot match the explanatory suppleness of even a mediocre novel.

- *Data creates bigger haystacks.* This is a point Nassim Taleb, the author of "Antifragile," has made. As we acquire more data, we have the ability to find many, many more statistically significant correlations. Most of these correlations are spurious and deceive us when were trying to understand a situation. Falsity grows exponentially the more data we collect. The haystack gets bigger, but the needle we are looking for is still buried deep inside.

- *Big data has trouble with big problems.* If you are trying to figure out which e-mail produces the most campaign contributions, you can do a randomized control experiment. But let's say you are trying to stimulate an economy in a recession. You don't have an alternate society to use as a control group. For example, we've had huge debates over the best economic stimulus, with mountains of data, and as far as I know not a single major player in this debate has been persuaded by data to switch sides.

- *Data favors mêmes over masterpieces.* Data analysis can detect when large numbers of people take an instant liking to some cultural product. But many important (and profitable) products are hated initially because they are unfamiliar.
- *Data obscures values.* I recently saw an academic book with the excellent title, "Raw Data Is an Oxymoron." One of the points was that data is never raw; its always structured according to somebodys predispositions and values. The end result looks disinterested, but, in reality, there are value choices all the way through, from construction to interpretation.

2 Model Components

A mathematical model has three main components: Decision Variables, Objective Function and Constraints.

2.1 Decision Variables

Decision variables capture the level of activities that the model studies. Decision makers have some freedom (subject to Constraints, see below) to assign numerical values to decision variables. For example, number of bolts (screws) produced in a week, denoted by B (S), is a common decision variable at machining plants. Letting, say, $B = 5000$ and $S = 7200$, we specify that 5000 bolts and 7200 screws are produced in a week. These activity levels of 5000 and 7200 specify a (production) plan over a week. The plan is not as detailed as specifying what to do every day. We can say that daily activity levels are abstracted out as they are agregated into weekly levels to facilitate computability. Solving a mathematical model means finding these numerical values for decision variables to minimize or maximize an *objective function* in the presence of *constraints*.

2.2 Objective Function

With mathematical models, we wish to maximize or minimize a quantity such as cost, profit, risk, net present value, number of employees, customer satisfaction, etc. The quantity we wish to maximize or minimize is known as *objective* (*function*). We say objective function to highlight the fact that objective is a function of decision variables.

Deciding on the correct objective in practical situations is not trivial. At one extreme there may be no clear objectives, at the other there may be multiple objectives. Multiple objectives, although possible in the case of a single decision maker, often arise with multiple decision makers. Reconciliation, weighing, or demotion of all but one of these objectives to contraints are among the methods to end up with a single objective. This process of honing down to a single objective involves discussions between the developers of the formulation and users of the formulation and it the takes place before formulation starts. The users must check and approve the final objective; a wrong objective can be worse than no objective at all. Sometimes developers may push for objectives that they are familiar with from past engagements or that are easier to formulate, but neither of these should be a concern for the users.

There are no recipes for the process of coming up with the correct objective. However, one develops a feeling for this process by studying various examples.

Suppose that the machining plant mentioned above makes a profit of \$13 from every 1000 bolts and \$15 from every 1000 screws. Then the objective function of maximizing weekly profit is:

$$\text{Objective Function :} \quad \max 0.013B + 0.015S.$$

What are the units of this objective function? Every time you write an objective function or a constraint, check the units.

2.3 Constraints

Constraints represent the limitations such as available capacity, daily working hours, raw material availability, etc. Sometimes constraints are also used to represent relationships between decision variables.

Suppose that the machining plant has 120 (with 3 lathes) hours of turning capacity per week. It has 36 hours of grinding capacity per week. Also two people work halftime and one person works fulltime for bolt and screw production making available number of manpower capacity 80 hours per week. Table below lays out turning, grinding and manpower hours needed to produce 1000 bolts and 1000 screws.

Activity	# of hrs required per 1000 bolts	# of hrs required per 1000 screws
Turning	3	4
Grinding	2	1
Manpower	1	3

If we produce B many bolts and S many screws, we need $3B/1000$ turning hours for bolts and $4S/1000$ turning hours for screws. Total turning hours needed is $3B/1000 + 4S/1000$, which has to be less than 120 hours available for turning per week:

$$\text{Turning Constraint}: \; 3B/1000 + 4S/1000 \leq 120.$$

Similarly we can write down two other constraints one for grinding capacity and one for manpower capacity:

$$\text{Grinding Constraint}: \; 2B/1000 + S/1000 \leq 36.$$

$$\text{Manpower Constraint}: \; B/1000 + 3S/1000 \leq 80.$$

Since both the numbers of bolts and screws are nonnegative numbers, we also add:

$$\text{Nonnegativity Constraints}: \; B \geq 0 \; , \; S \geq 0.$$

We obtain a formulation for the machining plant by putting the objective function and four constraints together. As an afterthought, do you think we really need the Nonnegativity Constraints? (Hint: Would you think removing Nonnegativity Constraints changes the solution?) We will revisit this question in much more detail.

Another instructive exercise is reformulating the machine plant problem after letting B and S be the number of bolts and screws in thousands. This is known as scaling a model (variables). Scaling can improve the accuracy of solution techniques but this is outside the scope of this note.

3 Linear Programming Assumptions

In the machining plant example above, a linear programming formulation is obtained with some taciturn assumptions. These assumptions are stated and clarified below. If you have not thought about these assumptions until now, you might find them very natural in the machining plant example. They are not so natural in the context of other examples.

1. Proportionality: Contribution of each activity to the objective function and a constraint is proportional to the level of that activity. For example, if 1000 bolts require 3 turning hours, 100 bolts require 0.3 hours and 2000 bolts require 6 hours. This assumption fails when there is (dis)economies of scale.

2. Additivity: Individual contribution of different activities can be summed up to obtain an objective function and constraints. For example, turning constraint is obtained by summing turning hours required by bolts and screws. This assumption fails when activities are not independent. For xample, for two synergistic activities, a higher level of one activity makes another activity simpler and/or less costly.

3. Certainty: Each parameter in the formulation is known for sure.

4. Divisibility: Decision variables can take integer as well as fractional values.

Without Proportionality or Additivity assumptions, we have Nonlinear Programs. Without Certainty assumption, we have Stochastic Programs. Without Divisibility assumption, we have Integer Programs.

3.1 Linear function/inequality/equality

In this section, we see some examples of linear and nonlinear objective functions and constraints. We start first from the definition of a linear function.

DEFINITION 1 (Linear function). *A function f, whose arguments (variables) are $(x_1, x_2, \dots, x_n)$, is linear if it can be written as*

$$f(x_1, x_2, \dots, x_n) \quad = \quad b + c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

by using constants $b, c_1, c_2, \dots, c_n$.

In this definition of the linear function, it is very important to distinguish between arguments of a function and the constants used to express the function. The following example should help with this distinction.

EXAMPLE 3. *Is $f = y_1 y_2 + y_1 z_1 + y_2 z_2$ a linear function? You cannot answer this question because the arguments of the function are not specified. Basically, the question is not well stated. There are at least three interesting cases:*

Case i) *If the arguments of the function are (z_1, z_2) then $f(z_1, z_2) = y_1 y_2 + y_1 z_1 + y_2 z_2$* **is** *a linear function of (z_1, z_2) because it can be written as*

$$f(z_1, z_2) = \underbrace{y_1 y_2}_{=:b} + \underbrace{y_1}_{=:c_1} \underbrace{z_1}_{=:x_1} + \underbrace{y_2}_{=:c_2} \underbrace{z_2}_{=:x_2} = b + c_1 x_1 + c_2 x_2.$$

Case ii) *If the arguments of the function are (y_1, y_2) then $f(y_1, y_2) = y_1 y_2 + y_1 z_1 + y_2 z_2$* **is not** *a linear function of (y_1, y_2) because it contains the multiplication of arguments y_1 and y_2. This reasoning will imply that any set of arguments that include (y_1, y_2) will lead to a nonlinear function. For example, $f(y_1, y_2, x_1)$, $f(y_1, y_2, x_2)$, $f(y_1, y_2, x_1, x_2)$ are all nonlinear on account of the term $y_1 y_2$.*

Case iii) *If the arguments of the function are (z_1, z_2, y_1) then $f(z_1, z_2, y_1) = y_1 y_2 + y_1 z_1 + y_2 z_2$* **is not** *a linear function of (z_1, z_2, y_1) because of terms $y_1 z_1$ annd $y_2 z_2$. Similarly, we can see that $f(z_1, z_2, y_2) = y_1 y_2 + y_1 z_1 + y_2 z_2$* **is not** *a linear function.*

Having seen the example above, we conclude that we must identify the arguments (variables) of a function before calling it linear or nonlinear. In an example where f is abstract, the example must state what the variables are. In an example from real-life, the analyst must have a sense of what the variables are. The next example illustrates a real-life case.

Example 4 (Payroll at a restaurant). *A fast food restaurant pays workers at the rate of minimum wage of \$6 per hour. The restaurant employs two workers Felix and Alex on a part-time basis; in a week, they work as much as required by the restaurant manager and they are paid proportional to their working hours. Is the total weekly payments made to Felix and Alex a linear function of their working hours?*

To answer this question formally, let h_F and h_A be the weekly working hours of Felix and Alex respectively. Then let g be the total weekly payment. Clearly, the payment is a function of h_F and h_A, so $g(h_F, h_A) = 6h_F + 6h_A$. The function $g(h_F, h_A)$ is linear in its arguments h_F and h_A because we can write it as

$$g(h_F, h_A) = \underbrace{0}_{=:b} + \underbrace{6}_{=:c_1}\underbrace{h_F}_{=:x_1} + \underbrace{6}_{=:c_2}\underbrace{h_A}_{=:x_2} = b + c_1x_1 + c_2x_2.$$

The payroll example above has a linear cost. In real life, many costs will be linear in their arguments. This is an artifact of the fact that accounting systems cost out activities proportional to the activity levels. However, there are some examples in accounting systems when the costs are not linear.

Example 5 (Payroll at a restaurant with overtime). *Continuing with the payroll example at the restaurant, suppose that workers are paid at the rate of \$8 per hour if they work overtime beyond 40 hours per week while the regular time pay is \$6 per hour as before. We can once more check to see if the new payment function is linear.*

$$g(h_F, h_A) = \left\{ \begin{array}{ll} 6h_F + 6h_A & \textit{if } h_F \le 40,\ h_A \le 40 \\ 6*40 + 8(h_F - 40) + 6h_A & \textit{if } h_F \ge 40,\ h_A \le 40 \\ 6h_F + 6*40 + 8(h_A - 40) & \textit{if } h_F \le 40,\ h_A \ge 40 \\ 6*40 + 8(h_F - 40) + 6*40 + 8(h_A - 40) & \textit{if } h_F \ge 40,\ h_A \ge 40 \end{array} \right\}$$

Since $6h_F \ge 6*40 + 8(h_F - 40)$ for $h_F \le 40$ and $6h_F \le 6*40 + 8(h_F - 40)$ for $h_F \ge 40$, we can rewrite the payment function as

$$g(h_F, h_A) = \left\{ \begin{array}{ll} \max\{6h_F, 6*40 + 8(h_F - 40)\} + 6h_A & \textit{if } h_A \le 40 \\ \max\{6h_F, 6*40 + 8(h_F - 40)\} + 6*40 + 8(h_A - 40) & \textit{if } h_A \ge 40 \end{array} \right\}.$$

Since $6h_A \ge 6*40 + 8(h_A - 40)$ for $h_A \le 40$ and $6h_A \le 6*40 + 8(h_A - 40)$ for $h_A \ge 40$, we can rewrite the payment function also as

$$g(h_F, h_A) = \max\{6h_F, 6*40 + 8(h_F - 40)\} + \max\{6h_A, 6*40 + 8(h_A - 40)\}.$$

Since the payment function has max *terms, it is not linear. However, it is possible to convert a maximum into inequalities in certain situations; for example see Section 7.*

Up to now, we have seen functions including simple forms like addition, multiplication and maximum. Still, not all of these functions are linear. Now, we can see some examples which include more complicated nonlinear terms like power functions, trigonometric functions, exponential functions, etc. Next example deals with these functions.

Example 6. *Power functions:*
a) $f(y_1) = y_1^3$ **is not** *linear in* y_1.
b) $f(y_1, y_2) = \sqrt{y_1^2 + y_2^2}$ **is not** *linear in* (y_1, y_2).
c) $f(y_1) = \sqrt{y_1^2 + y_2^2}$ **is not** *linear in* y_1.
d) $f(y_1) = 3^{y_1}$ **is not** *linear in* y_1.
e) $f(y_1, y_2) = 3^{y_1} + y_1y_2$ **is not** *linear in* (y_1, y_2).
f) $f(y_2) = 3^{y_1} + y_1y_2$ **is** *linear in* y_2 *because*

$$f(y_2) = \underbrace{3^{y_1}}_{=:b} + \underbrace{y_1}_{=:c_1}\underbrace{y_2}_{=:x_1} = b + c_1x_1.$$

Trigonometric functions:
a) $f(y_1, y_2) = \sin(y_1 + y_2)$ **is not** *linear in* (y_1, y_2).
b) $f(y_1, y_2, y_3) = \cos(y_1 + y_2) + \pi y_3 - y_1 y_2$ **is not** *linear in* (y_1, y_2, y_3).
c) $f(y_1, y_3) = \cos(y_1 + y_2) + \pi y_3 - y_1 y_2$ **is not** *linear in* (y_1, y_3).
d) $f(y_3) = \cos(y_1 + y_2) + \pi y_3 - y_1 y_2$ **is** *linear in* y_3 *because*

$$f(y_3) = \underbrace{\cos(y_1 + y_2) - y_1 y_2}_{=:b} + \underbrace{\pi}_{=:c_1} \underbrace{y_3}_{=:x_1} = b + c_1 x_1.$$

e) $f(y_1, y_2) = \dfrac{1 - \sin^2(y_1)}{\sin^2(y_1)} y_2 + y_1 \dfrac{\tan(y_2)}{\sin(y_2)}$ **is not** *linear in* (y_1, y_2).
f) $f(y_1, y_2) = \dfrac{1 - \sin^2(y_1)}{\cos^2(y_1)} y_2 + y_1 \dfrac{\tan(y_2)}{\sin(y_2)} cos(y_2)$ **is** *linear in* (y_1, y_2) *because*

$$f(y_1, y_2) = \underbrace{\frac{1 - \sin^2(y_1)}{\cos^2(y_1)}}_{=1} y_2 + y_1 \underbrace{\frac{\tan(y_2)}{\sin(y_2)/\cos(y_2)}}_{=1} = \underbrace{0}_{=:b} + \underbrace{1}_{=:c_1} * \underbrace{y_2}_{=:x_1} + \underbrace{1}_{=:c_2} * \underbrace{y_1}_{=:x_2} = b + c_1 x_1 + c_2 x_2.$$

As seen in the last example, knowledge of the identities involving trigonometric functions is useful to simplify expressions. Above we have used $\cos^2(y_1) = 1 - \sin^2(y_1)$ *and* $\tan(y_2) = \sin(y_2)/\cos(y_2)$.

Exponential functions use the constant $e = 2.71828$*, which is also known as Euler's number. The inverse of exponential is logarithm:*
a) $f(y_1, y_2) = e^{y_1 + y_2} = e^{y_1} e^{y_2}$ **is not** *linear in* (y_1, y_2).
b) $f(y_1, y_2) = e^{y_1 + y_2} + \log(y_1 + y_2)$ **is not** *linear in* (y_1, y_2).
c) $f(y_1, y_2) = \log(y_1 + y_2)$ **is not** *linear in* (y_1, y_2).
d) $f(y_1, y_2) = \log(y_1 y_2) = \log(y_1) + \log(y_2)$ **is not** *linear in* (y_1, y_2).
e) $f(y_1, y_2) = e^{\log y_1} y_2 + e y_2 + y_1^2 y_2 - \tan(y_1)$ **is not** *linear in* (y_1, y_2).
f) $f(y_2) = e^{\log y_1} y_2 + e y_2 + y_1^2 y_2 - \tan(y_1)$ **is** *linear in* y_2 *because*

$$f(y_2) = (\underbrace{e^{\log y_1}}_{=y_1} + e + y_1^2) y_2 - \tan(y_1) = \underbrace{-\tan(y_1)}_{=:b} + \underbrace{(y_1 + e + y_1^2)}_{=:c_1} \underbrace{y_2}_{=:x_1} = b + c_1 x_1.$$

As we see from the last example, power functions, trigonometric functions, exponential functions often lead to nonlinear functions unless there is an identity available to simplify the expressions. An interesting example of a power and trigonometric function is drawn in three dimensions in Figure 1. Although nonlinearity of that figure is interesting, the linear programs cannot accommodate any nonlinearity.

Having discussed the linearity of functions, we now focus on linear inequalities. The definition of linear inequality is derived from the linear functions.

Definition 2 (Linear inequality). *The expression*

$$f(x_1, x_2, \ldots, x_n) \quad \leq \quad b$$

is a linear inequality provided that $f(x_1, x_2, \ldots, x_n)$ *is a linear function of* $(x_1, x_2, \ldots, x_n)$ *and* b *is a constant.*

Since $-f(x_1, x_2, \ldots, x_n)$ is linear if and only if $f(x_1, x_2, \ldots, x_n)$ is linear, we can write an inequality of the form

$$f(x_1, x_2, \ldots, x_n) \quad \geq \quad b$$

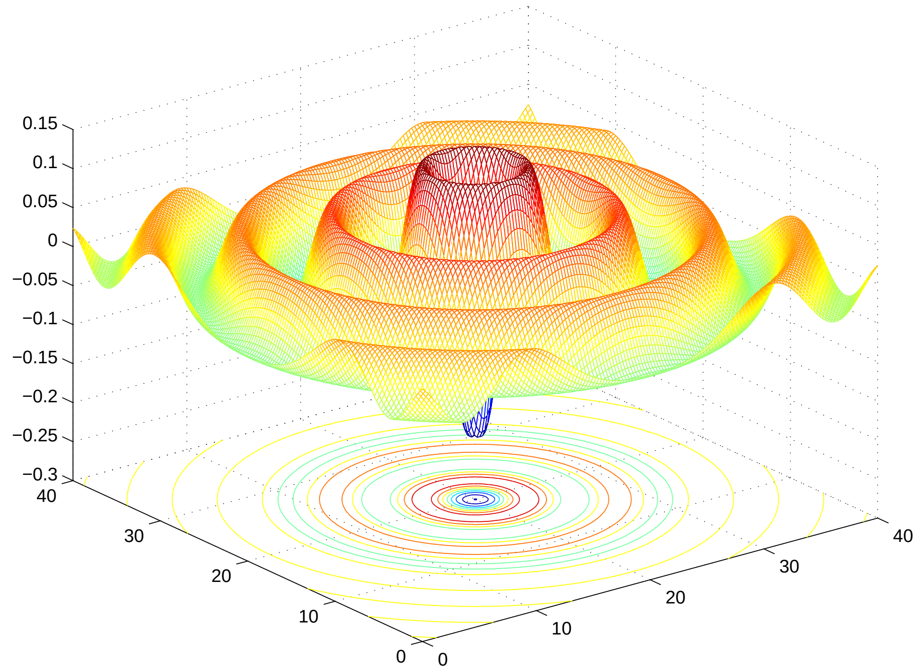

Figure 1: "Flowers" are not allowed in linear programs. Equation of the "flower" surface centered at $(x_1 = 20, x_2 = 20)$ is $x_3(\sqrt{(x_1-20)^2+(x_2-20)^2+4}) = \sin(\sqrt{(x_1-20)^2+(x_2-20)^2+4})$

as

$$-f(x_1, x_2, \ldots, x_n) \quad \leq \quad -b =: b'$$

where b' is a constant. As a result we obtain the next remark.

Remark 1. *The expression*

$$f(x_1, x_2, \ldots, x_n) \quad \geq \quad b$$

is a linear inequality provided that $f(x_1, x_2, \ldots, x_n)$ is a linear function of $(x_1, x_2, \ldots, x_n)$ and b is a constant.

Since

$$f(x_1, x_2, \ldots, x_n) \quad = \quad b$$

can always be written as

$$f(x_1, x_2, \ldots, x_n) \leq b \ \text{ and } \ f(x_1, x_2, \ldots, x_n) \geq b,$$

we consider $f(x_1, x_2, \ldots, x_n) = b$ two linear inequalities that can be included in a linear program. These observations can be summarized in the next remark.

Remark 2. *The inequalities and equalities*

$$f(x_1, x_2, \ldots, x_n) \leq b, \ f(x_1, x_2, \ldots, x_n) \geq b \ \textit{ and } \ f(x_1, x_2, \ldots, x_n) = b.$$

can be included among the constraints of a linear program provided that $f(x_1, x_2, \ldots, x_n)$ is a linear function of $(x_1, x_2, \ldots, x_n)$ and b is a constant.

Equipped with the last remark, now we can look at some examples to see which can be included in a linear program.

Example 7. *For each of the inequality determine if it is a linear inequality/equality in variables $x_1, x_2, \ldots, x_n$. If the original inequality is not linear, can you find equivalent linear inequalities/equalities to put into a linear program?*

a) $-3x_1 - 5x_2 + 22 \geq 18$.

This is a linear inequality and can be written as $-3x_1 - 5x_2 \geq -4$ *where the left-hand side is a linear function.*

b) $3y_1x_1 + 2x_2 + y_2^2 \leq 1 - y_1$.

This is a linear inequality and can be written as $3y_1x_1 + 2x_2 \leq 1 - y_1 - y_2^2$ *where the left-hand side is a linear function.*

c) $2x_1^2 - 8 \leq 0$.

This can be written as $x_1^2 \leq 4$ *but the left-hand side is not a linear function. However,* $x_1^2 \leq 4$ *if and only if* $-2 \leq x_1 \leq 2$. *In this case, the original inequality is not linear but it is equivalent to linear inequalities:* $x_1 \leq 2$ *and* $x_1 \geq -2$.

d) $x_1x_2 - 3x_1 - x_2 + 3 = 0$.

This can be written as $(x_1 - 1)(x_2 - 3) = 0$. *Then, either* $x_1 = 1$ **or** $x_2 = 3$. *However, we cannot put* **either-or** *into a linear program. Compare this with two inequalities* $x_1 \leq 2$ **and** $x_1 \geq -2$ *used to represent* $x_1^2 \leq 4$. *The set of the inequalities/equalities constituting the constraints of a linear program by definition are satisfied all together, so the logical operator connecting constraints is always* **and** *not* **either-or**. *That is why* **either-or** *type constraints cannot be included as constraints of a linear program.*

e) $\dfrac{12x_1}{3x_1 - x_2} \leq \pi$.

The original inequality is not linear but it can be rewritten as $(12 - 3\pi)x_1 + \pi x_2 \leq 0$ *where the left-hand side is a linear function. In this case, the original inequality is not linear but it is equivalent to linear inequality* $(12 - 3\pi)x_1 + \pi x_2 \leq 0$.

f) $e^{3x_1 + 5x_2} \geq 12$.

The original inequality is not linear but it is equivalent to $3x_1 + 5x_2 \geq \log 12$ *where the left-hand side is a linear function. In this case, the original inequality is not linear but it is equivalent to linear inequality* $3x_1 + 5x_2 \geq \log 12$.

g) $x_1 e^{x_2} \geq 12$.

The original inequality is not linear. Taking the logarithm of both sides of the inequality, we obtain $\log x_1 + x_2 \geq \log 12$. *On account of the* $\log x_1$, *this inequality is not linear either.*

h) $3^{x_1} 5^{-x_2} \leq 1$.

The original inequality is not linear. Taking the logarithm of both sides of the inequality, we obtain $(\log 3)x_1 - (\log 5)x_2 \leq 0$ *where the left-hand side is a linear function.*

i) $\dfrac{x_1^2}{4} + \dfrac{x_2^2}{9} \leq 1$.

This inequality is not linear. It actually specifies the area inside an ellipse centered at $(x_1 = 0, x_2 = 0)$ *intersecting the* x_1 *axis at -2 and 2, and intersecting the* x_2 *axis at -3 and 3.*

j) $\max\{x_1, -x_1 + 7x_2, x_2^3\} \leq 8$.

The original inequality is not linear. However, the max can be split into three inequalities: $x_1 \leq 8$, $-x_1 + 7x_2 \leq 8$ *and* $x_2 \leq 2$. *These are all linear inequalities connected by* **and** *logical operator, so they can be included in a linear program to replace the original inequality.*

k) $\max\{x_1, -x_1 + 7x_2, x_2^3\} \geq 8$.

The original inequality is not linear. However, the max can be split into three inequalities: $x_1 \geq 8$ *or* $-x_1 + 7x_2 \geq 8$ *or* $x_2 \geq 2$. *These are all linear inequalities. They are connected by* **or** *logical operator, so they cannot be included in a linear program.*

l) $\min\{2x_1, 3x_2\} \geq 8$.
The original inequality is not linear. However, the min can be split into two inequalities: $2x_1 \geq 8$ *and* $3x_2 \geq 8$*. These are both linear inequalities connected by* **and** *logical operator, so they can be included in a linear program to replace the original inequality.*
m) $\min\{2x_1, 3x_2\} \leq 8$.
The original inequality is not linear. However, the min can be split into two inequalities: $2x_1 \leq 8$ *or* $3x_2 \leq 8$*. These are both linear inequalities. They are connected by* **or** *logical operator, so they cannot be included in a linear program.*
n) $\min\{2x_1, 3x_2\} \geq \max\{10 - x_1, 6 - x_2\}$.
The original inequality is not linear. However, the min and max can be split into four inequalities: $2x_1 \geq 10 - x_1$*,* $2x_1 \geq 6 - x_2$*,* $3x_2 \geq 10 - x_1$ *and* $3x_2 \geq 6 - x_2$*. These are all linear inequalities connected by* **and** *logical operator, so they can be included in a linear program to replace the original inequality.*

With the fourteen different original inequalities discussed in the last example, one should gain a reasonable appreciation of the linearity of inequalities and equalities. Another example on this topic is also provided in the exercises section below.

4 A Production Planning Problem

Suppose a production manager is responsible for scheduling the monthly production levels of a certain product for a planning horizon of twelve months. For planning purposes, the manager was given the following information:

- The total demand for the product in month j is d_j, for $j = 1, 2, \ldots, 12$. These could either be targeted values or be based on forecasts.
- The cost of producing each unit of the product in month j is c_j (dollars), for $j = 1, 2, \ldots, 12$. There is no setup/fixed cost for production.
- The inventory holding cost per unit for month j is h_j (dollars), for $j = 1, 2, \ldots, 12$. These are incurred at the end of each month.
- The production capacity for month j is m_j, for $j = 1, 2, \ldots, 12$.

The manager's task is to generate a production schedule that minimizes the total production and inventory-holding costs over this twelve-month planning horizon.

To facilitate the formulation of a linear program, the manager decides to make the following simplifying assumptions:

1. There is no initial inventory at the beginning of the first month.
2. Units scheduled for production in month j are immediately available for delivery at the beginning of that month. This means in effect that the production rate is infinite.
3. Shortage of the product is not allowed at the end of any month.

To understand things better, let us consider the first month. Suppose, for that month, the planned production level equals 100 units and the demand, d_1, equals 60 units. Then, since the initial inventory is 0 (Assumption 1), the ending inventory level for the first month would be 0+100-60=40 units. Note that all

100 units are immediately available for delivery (Assumption 2); and that given $d_1 = 60$, one must produce no less than 60 units in the first month, to avoid shortage (Assumption 3). Suppose further that $c_1 = 15$ and $h_1 = 3$. Then, the total cost for the first month can be computed as: $15 \cdot 100 + 3 \cdot 60 = 1380$ dollars.

At the start of the second month, there would be 40 units of the product in inventory, and the corresponding ending inventory can be computed similarly, based on the initial inventory, the scheduled production level, and the total demand for that month. The same scheme is then repeated until the end of the entire planning horizon.

4.1 Decision Variables

The manager's task is to set a production level for each month. Therefore, we have twelve decision variables:

- x_j = the production level for month j, $j = 1, 2, \ldots, 12$.

EXAMPLE 8. *Suppose $d_1 = 60$, $d_2 = 90$ and $d_3 = 100$ and the manager sets $x_1 = 100$, $x_2 = 80$ and $x_3 = 70$. The inventory at the end of the first month is 40=100-60=$x_1 - d_1$. The inventory at the end of the second month is 30=100-60+80-90=$x_1 - d_1 + x_2 - d_2$. The inventory at the end of the third month is 0=100-60+80-90+70-100=$x_1 - d_1 + x_2 - d_2 + x_3 - d_3$.*

4.2 Objective Function

Consider the first month again. From the discussion above, we have:

- The production cost equals $c_1 x_1$.
- The inventory-holding cost equals $h_1(x_1 - d_1)$, provided that the ending inventory level, $x_1 - d_1$, is nonnegative.

Therefore, the total cost for the first month equals $c_1 x_1 + h_1(x_1 - d_1)$.
For the second month, we have:

- The production cost equals $c_2 x_2$.
- The inventory-holding cost equals $h_2(x_1 - d_1 + x_2 - d_2)$, provided that the ending inventory level, $x_1 - d_1 + x_2 - d_2$, is nonnegative. This follows from the fact that the starting inventory level for this month is $x_1 - d_1$, the production level for this month is x_2, and the demand for this month is d_2.

Therefore, the total cost for the second month equals $c_2 x_2 + h_2(x_1 - d_1 + x_2 - d_2)$.

Continuation of this argument yields that:

- The total production cost for the entire planning horizon equals

$$\sum_{j=1}^{12} c_j x_j = c_1 x_1 + c_2 x_2 + \cdots + c_{12} x_{12} \,,$$

  where we have the standard summation notation.

- The inventory-holding cost at the end of first month is $h_1(x_1 - d_1)$. At the end of the second month, it is

$$h_2(x_1 - d_1 + x_2 - d_2) = h_2 \sum_{k=1}^{2}(x_k - d_k).$$

  At the end of the third month, it is

$$h_2(x_1 - d_1 + x_2 - d_2 + x_3 - d_3) = h_3 \sum_{k=1}^{3}(x_k - d_k).$$

  At the end of the jth month, it is

$$h_2(x_1 - d_1 + x_2 - d_2 + \cdots + x_j - d_j) = h_j \sum_{k=1}^{j}(x_k - d_k).$$

  The total inventory-holding cost for the entire planning horizon equals

$$\begin{aligned}
\sum_{j=1}^{12} h_j \left[\sum_{k=1}^{j}(x_k - d_k)\right] &= h_1 \left[\sum_{k=1}^{1}(x_k - d_k)\right] + h_2 \left[\sum_{k=1}^{2}(x_k - d_k)\right] + \cdots + h_{12} \left[\sum_{k=1}^{12}(x_k - d_k)\right] \\
&= h_1 \left[x_1 - d_1\right] + h_2 \left[(x_1 - d_1) + (x_2 - d_2)\right] + \ldots \\
&\quad + h_{12} \left[(x_1 - d_1) + (x_2 - d_2) + \cdots + (x_{12} - d_{12})\right].
\end{aligned}$$

Since our goal is to minimize the total production and inventory-holding costs, the objective function can now be stated as

$$\text{Min} \quad \sum_{j=1}^{12} c_j x_j + \sum_{j=1}^{12} h_j \left[\sum_{k=1}^{j}(x_k - d_k)\right].$$

4.3 Constraints

Since the production capacity for month j is m_j, we require

$$x_j \le m_j$$

for $j = 1, 2, \ldots, 12$; and since shortage is not allowed (Assumption 3), we require

$$\sum_{k=1}^{j}(x_k - d_k) \ge 0$$

for $j = 1, 2, \ldots, 12$. This results in a set of 24 functional constraints. Of course, being production levels, the x_j's should be nonnegative.

4.4 LP Formulation

In summary, we have arrived at the following formulation:

$$\begin{array}{lrcll}
\text{Min} & \multicolumn{4}{l}{\sum_{j=1}^{12} c_j x_j + \sum_{j=1}^{12} h_j \left[\sum_{k=1}^{j}(x_k - d_k)\right]} \\
\textit{Subject to:} & & & & \\
& x_j & \le & m_j & for\ \ j = 1, 2, \ldots, 12 \\
& \sum_{k=1}^{j}(x_k - d_k) & \ge & 0 & for\ \ j = 1, 2, \ldots, 12 \\
& \multicolumn{4}{l}{x_j \ge 0 \quad for\ \ j = 1, 2, \ldots, 12\,.}
\end{array}$$

This is a linear program with 12 decision variables, 24 functional constraints, and 12 nonnegativity constraints. In an actual implementation, we need to replace the c_j's, the h_j's, the d_j's, and the m_j's with explicit numerical values.

EXAMPLE 9. *An analyst is in charge of production planning at plant for the next 4 months. She estimates demands to be $d_1 = 700$, $d_2 = 300$, $d_3 = 500$ and $d_4 = 600$. The production csot is $c = 50$ per unit and $h = 5$ per unit and per month. The plant has ample capacity, i.e., $m_j = \infty$ in month j. Therefore there is no capacity constraint. The analyst comes up the following linear program to minimize costs.*

$$\begin{aligned} &\textit{Min} && 50(x_1+x_2+x_3+x_4) + 5(x_1-700+x_1-700+x_2-300+x_1-700+x_2-300+x_3-500 \\ & && \qquad +x_1-700+x_2-300+x_3-500+x_4-600) \\ &\textit{Subject to}: && \\ & && x_1 \geq 700;\ x_1+x_2 \geq 1000;\ x_1+x_2+x_3 \geq 1500;\ x_1+x_2+x_3+x_4 \geq 2100, \\ & && x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

It shouold be fairly clear how 1000, 1500 and 2100 are found.

4.5 An Alternative Formulation for Production Planning

In the above formulation, the expression for the total inventory-holding cost in the objective function involves a nested sum, which is rather complicated. Notice that for any given j, the inner sum in that expression, $\sum_{k=1}^{j}(x_k - d_k)$, is simply the ending inventory level for month j. This motivates the introduction of an additional set of decision variables to represent the ending inventory levels. Specifically, let

- y_j = the ending inventory level for month j, j = 1, 2, ..., 12;

Then, the objective function can be rewritten in the following simpler-looking form:

$$\text{Min} \quad \sum_{j=1}^{12} c_j x_j + \sum_{j=1}^{12} h_j y_j \ .$$

With these new variables, the no-shortage constraints also simplify to $y_j \geq 0$ for j = 1, 2, ..., 12. However, we now need to introduce a new set of constraints to "link" the x_j's and the y_j's together.
Consider the first month again. Denote the initial inventory level as y_0; then, by assumption 1, we have $y_0 = 0$. Since the production level is x_1 and the demand is d_1 for this month, we have $y_1 = y_0 + x_1 - d_1$. Continuation of this argument shows that for j = 1, 2, ..., 12,

$$y_j = y_{j-1} + x_j - d_j$$

and these relations should appear as constraints to ensure that the y_j's indeed represent ending inventory levels. We have, therefore, arrived at the following new formulation:

$$\begin{aligned} &\text{Min} && \sum_{j=1}^{12} c_j x_j + \sum_{j=1}^{12} h_j y_j \\ &\textit{Subject to}: && \\ & && \begin{array}{lcll} x_j & \leq & m_j & \textit{for } j = 1,2,\ldots,12 \\ y_j & = & y_{j-1} + x_j - d_j & \textit{for } j = 1,2,\ldots,12 \\ x_j & \geq & 0 & \textit{for } j = 1,2,\ldots,12 \\ y_j & \geq & 0 & \textit{for } j = 1,2,\ldots,12. \end{array} \end{aligned}$$

which is a linear program with 24 decision variables, 24 functional constraints, 12 equality contraints, and 24 nonnegative variables.

Although there are twice as many decision variables in the new formulation, both formulations have the same number of functional constraints. Many modern LP solvers run pre-solve algorithms to detect and eliminate equalities before starting the solution algorithm. Even if you input the alternative formulation into your solver, the solver may convert it to the original formulation before starting the solution procedure. This should not be alarming as two formulations are equivalent. In general, it is not uncommon to have several equivalent formulations of the same problem.

4.6 Remarks

- If Assumption 1 is relaxed, so that the initial inventory level is not necessarily zero, we can simply set y_0 to whatever given value.
- In our formulation, we assumed that there is no production delay (Assumption 2). This assumption can be easily relaxed. Suppose instead there is a production delay of one month; that is, the scheduled production for month j, x_j, is available only after a delay of one month, i.e., in month $j+1$. Then, in the alternative formulation, we can simply replace the constraint $y_j = y_{j-1} + x_j - d_j$ by

$$y_j = y_{j-1} + x_{j-1} - d_j$$

  (with $x_0 = 0$), for $j = 1, 2, \ldots, 12$. Of course, for the first month, the given value of y_0 must be no less than d_1; otherwise, the resulting LP will not have any solution.
- Assumption 3 can also be relaxed. If shortages are allowed, we can simply remove the nonnegativity requirements for the inventory, and introduce a shortage penalty cost of, say, p_j per unit of shortage at the end of month j.

5 A Blending Problem

A salad dressing supplier to major DFW area restaurants is considering using LP for its blending problem. This dressing (market value of $400/ton) is manufactured by refining raw oils and blending them together. Five types of oils come in two categories. Olive oils: Olive 1 and Olive 2. Corn oils: Corn 1, Corn 2 and Corn 3. Oil prices ($ per ton) for the coming three months are given as follows:

	Olive 1	Olive 2	Corn 1	Corn 2	Corn 3
Oct	280	390	110	180	130
Nov	290	400	90	190	130
Dec	310	430	100	200	130

During refining process olive and corn oils can not be mixed. The dressing supplier dedicates a separate refining unit to each of the olive and corn oils. In any month, olive (corn) oil refining unit can process at most 190 (270) tons of oil. There are five tanks to store each type of oil separately, each tank has 300 tons of capacity. It is unhealthy to store refined oil. Storage costs per ton per month are $10 for all types of oil.

The hardness of salad dressings is regulated and has to be within 3 and 6. Generally hardness blends linearly and the hardness of the raw oils are given below:

Olive 1	Olive 2	Corn 1	Corn 2	Corn 3
3.1	2.4	7.2	5.8	6.1

We will formulate this blending problem to maximize supplier's profit.

5.1 Decision Variables

We are interested in quantities of raw oil bought, used and stored in each month. Let $O1B_i$ be the olive oil 1 bought in month i ($i = 1$ corresponds to Oct and so on), similarly define $O1U_i$ and $O1S_i$ as the olive oil 1 used in month i and stored at the end of month i. $O2B_i$, $O2U_i$ and $O2S_i$ refer to bought, used and stored olive oil 2. $C1B_i$, $C1U_i$ and $C1S_i$ refer to bought, used and stored corn oil 1 and so on. Also let D_i be the dressing produced and sold in month i.

Warning: Note that the first step in developing a formulation is clearly identifying decision variables. Without such an identification it is impossible to understand the notation in the objective function and the constraints.

5.2 Objective Function

We want to maximize the (profit = revenue - cost). Revenues are obtained by selling the dressing: $\sum_{i=1}^{3} 400 \cdot D_i$. Cost has two components: raw oil purchase costs and storage costs. Raw oil purchase cost: $280O1B_1 + 390O2B_1 + 110C1B_1 + 180C2B_1 + 130C3B_1 + 290O1B_2 + 400O2B_2 + 90C1B_2 + 190C2B_2 + 130C3B_2 + 310O1B_3 + 430O2B_3 + 100C1B_3 + 200C2B_3 + 130C3B_3$. Storage costs: $10(O1S_1 + O2S_1 + C1S_1 + C2S_1 + C3S_1 + O1S_2 + O2S_2 + C1S_2 + C2S_2 + C3S_2 + O1S_3 + O2S_3 + C1S_3 + C2S_3 + C3S_3)$.

5.3 Constraints

Storage transition constraints: $OjS_i = OjS_{i-1} + OjB_i - OjU_i$ for $j = 1, 2$ and $i = 1, 2, 3$. $CjS_i = CjS_{i-1} + CjB_i - CjU_i$ for $j = 1, 2, 3$ and $i = 1, 2, 3$. For simplicity assume that initially there are no oils on stock, i.e. OjS_0 and CjS_0 are both 0.

Storage tank capacity constraints: For olive oil storage tanks $OjS_i \leq 300$ for $i = 1, 2, 3$ and $j = 1, 2$. For corn oil storage tanks $CjS_i \leq 300$ for $i = 1, 2, 3$ and $j = 1, 2, 3$.

Refining capacity constraints: For olive oil refining $O1U_i + O2U_i \leq 190$ for $i = 1, 2, 3$. For corn oil refining $C1U_i + C2U_i + C3U_i \leq 270$ for $i = 1, 2, 3$.

Hardness constraints:

$$3 \leq \frac{3.1O1U_i + 2.4O2U_i + 7.2C1U_i + 5.8C2U_i + 6.1C3U_i}{D_i} \leq 6 \;\; for \; i = 1, 2, 3.$$

Manipulating above inequlity, we obtain the following inequalities:

$$3.1O1U_i + 2.4O2U_i + 7.2C1U_i + 5.8C2U_i + 6.1C3U_I - 6D_i \leq 0 \;\; for \; i = 1, 2, 3.$$

$$3.1O1U_i + 2.4O2U_i + 7.2C1U_i + 5.8C2U_i + 6.1C3U_I - 3D_i \geq 0 \;\; for \; i = 1, 2, 3.$$

Weight conservation constraint:

$$O1U_i + O2U_i + C1U_i + C2U_i + C3U_j - D_i = 0 \;\; for \; i = 1, 2, 3.$$

Nonnegativity constraints: $OjU_i \geq 0$ for $i = 1, 2, 3$ and $j = 1, 2$. $CjU_i \geq 0$ for $i = 1, 2, 3$ and $j = 1, 2, 3$. $D_i \geq 0$ for $i = 1, 2, 3$.

5.4 Remarks

- In the blending problem, we assumed that the customers purchase all the salad dressings produced. Now suppose that customers commit to purchase at least l_i and at most u_i tons of dressing in month i, modify the formulation accordingly. If there were no refining capacity constraints, would l_i be a redundant parameter (i.e. WLOG, could it be set to 0?), why?
- After solving the blending formulation, you realize that all the decision variables corresponding to oil storage at the end of month 3 are 0. Is this a coincidence? How would you justify this to dressing supplier? (Hint: think about the length of decision horizons.)

6 An Investment Problem

Suppose an investor has \$100 on Monday. At the start of **every** day of the week (Monday through Friday), the investor has the following investment opportunity available: If he invests x dollars on that day and matches that initial investment with $x/2$ dollars the next day, then he will receive a total return of $2x$ dollars on the third day. Thus, with a total investment of $1.5x$ dollars, the investor receives $2x$ dollars in two days, a gain of $0.5x$ dollars. The investor wishes to determine an investment schedule that maximizes his total cash on Saturday.

To facilitate the formulation of a linear program, the investor decides to make the following simplifying assumptions:

1. If an initial investment is not matched on the subsequent day, the initial investment is lost.
2. Any return that is due on any given day can be reinvested immediately.
3. Cash carried forward from one day to the next does not accrue interest.
4. Borrowing money is not allowed.

To understand things better, let us consider the following "naive" strategy: Begin with an investment of (2/3)100 dollars on Monday, while putting aside (1/3)100 dollars in anticipation of the necessary second installment on the next day (Assumption 1). On Tuesday, the investor executes the second installment and, consequently, he won't have any remaining cash to initiate a new investment. On Wednesday, the investor receives a total return of 2(2/3)100 dollars. This completes an investment cycle, during which the total amount invested in two installments (\$100) grew by a factor of 4/3. Suppose further that the investor immediately reinvests the yield he receives on Wednesday (Assumption 2) in the same manner as in the just-completed investment cycle. Then, a similar analysis shows that he will receive a total yield of $(4/3)(4/3)100 = 177.8$ dollars on Friday. At that point, since any new investment that starts on Friday won't mature until Sunday (a day too late), the investor simply carries this second yield into Saturday, resulting in a final cash position of 177.8 dollars.

How good is this naive strategy? While we managed to complete two full investment cycles, it seems uncomforting to watch cash sit idle from Friday to Saturday. This suggests that we might be able to do better. But how? Note that under the divisibility assumption, the set of possible strategies is a continuum, and hence it cannot even be enumerated. Thus, the task is challenging. To find out what is the best strategy, we now turn to an LP formulation of this problem.

6.1 Decision Variables

Since a new investment cycle can, potentially, be initiated at the start of each day, the investor needs to determine the magnitudes of these first installments. Therefore, let

- x_j = the amount of new investment at the beginning of Day j, $j = 1, 2, 3, 4$,

where Day 1 corresponds to Monday, Day 2 to Tuesday, and so on. There is no need to introduce x_5, why? We also assume that these decisions are to be made immediately **after** receiving yields (if any) from prior investments.

In principle, it seems sufficient (and it is) to work with these four decision variables alone: On Monday, we would invest x_1 dollars and carry a cash saving of $100 - x_1$ forward to Tuesday. On Tuesday, after executing a second installment of $x_1/2$ dollars, we would have $100 - x_1 - x_1/2$ dollars available for an allocation of a new investment and a new cash saving. A little bit of reflection should be convincing that the picture would become rather complex as we move further into future days. In particular, expressing the amount of cash saving at the end of each day as a function of the entire *history* of past decisions quickly becomes a difficult task.

From the previous production-planning example, we learned that it is sometimes desirable to introduce additional sets of decision variables for the purpose of simplifying the formulation task. Now, imagine yourself being at the start of a given day and ask: What actions do I need to take at this point? The answer is:

1. Cough up half of the amount of new investment (if any) that started in the previous day.
2. Initiate a new investment.
3. Carry the remaining cash (if any) forward to the next day.

Observe that these actions cannot be committed unless we know how much saving is being carried forward from the previous day; and that this information depends on the entire prior investment history in a complicated way. To circumvent this difficulty, it therefore seems desirable to define a new set of variables to represent the daily savings. Formally, let

- s_j = the amount of saving carried forward from Day j to Day $j + 1$, $j = 1, 2, 3, 4, 5$.

Note that we can think of the initial capital as a saving from Day 0 to Day 1; that is, let $s_0 = 100$. In summary, we have defined a total of 9 decision variables, four x_j's and five s_j's.

6.2 Objective Function

On Saturday (Day 6), there are two income streams; one is the yield from the investment cycle that started on Thursday and the other is the saving from Friday. The first contribution equals $2x_4$ and the second, s_5. It is important to realize that yields from all earlier investments are "implicitly captured" into these two terms. Thus, our objective is to:

$$\text{Max } 2x_4 + s_5.$$

Note the simplicity of this objective function. It is a consequence of our choice of the decision variables. In general, it is a good practice to conceptualize the formulation of the decision variables and the objective function jointly.

6.3 Constraints

In the alternative formulation of the production-planning problem, the production levels and the ending inventory levels were linked together via constraints like $y_j = y_{j-1} + x_j - d_j$ to ensure that they logically correspond to their intended interpretations. This can be viewed as "material" balancing. Here again, we need to ensure proper linkage between the daily new investments and the daily savings.

The basic idea is to balance the cash flow at the beginning of each day. For Day 1, we have $s_0 = 100$ dollars available; and this amount is apportioned into two parts: a new investment and a saving. Thus,

$$s_0 = x_1 + s_1.$$

For Day 2, we have s_1 dollars available; and this is apportioned into three parts: $0.5x_1$, x_2, and s_2. Thus,

$$s_1 = 0.5x_1 + x_2 + s_2.$$

At the beginning of Day 3, we receive a yield of $2x_1$, which is immediately available for reinvestment. It is therefore added into s_2; and the total amount is then divided into three parts as in Day 2. This leads to

$$2x_1 + s_2 = 0.5x_2 + x_3 + s_3.$$

Continuation of this argument yields

$$2x_2 + s_3 = 0.5x_3 + x_4 + s_4 \text{ for Day 4.}$$

$$2x_3 + s_4 = 0.5x_4 + s_5 \text{ for Day 5.}$$

Clearly, the x_j's must be nonnegative. To ensure that we never over spend, the s_j's are required to be nonnegative as well.

6.4 LP Formulation

$$\begin{array}{ll} \text{Max} & 2x_4 + s_5 \\ \text{Subject to:} & \end{array}$$

$$\begin{aligned} s_0 &= x_1 + s_1 \\ s_1 &= 0.5x_1 + x_2 + s_2 \\ 2x_1 + s_2 &= 0.5x_2 + x_3 + s_3 \\ 2x_2 + s_3 &= 0.5x_3 + x_4 + s_4 \\ 2x_3 + s_4 &= 0.5x_4 + s_5 \end{aligned}$$

$$x_j \geq 0 \text{ for } j = 1, 2, 3, 4 \text{ and } s_j \geq 0 \text{ for } j = 1, 2, 3, 4, 5.$$

This is a linear program with 9 decision variables, 5 functional constraints, and 9 nonnegativity constraints.

6.5 Remarks

1. It may be instructive to attempt to formulate this problem using the x_j's only. Give it a try; it would be quite messy.

2. If we did not realize that x_5 was unnecessary, then the fifth constraint would have come out as $2x_3 + s_4 = 0.5x_4 + x_5 + s_5$. Consider an investment strategy that prescribes, say, $x_5 = 5$ and $s_5 = 10$. We will argue that this strategy cannot be optimal. Observe that the variable x_5 appears only in the fifth constraint. Therefore, if we simply reset x_5 to 0 and s_5 to 15 (maintaining the sum $x_5 + s_5$ at 15), then the resulting new strategy will have a better objective function value (by how much?). This shows that it is never optimal to assign a positive value to x_5.

3. Relax Assumption 2: Suppose instead that there is a reinvestment delay of one day. This implies, for example, that the yield $2x_1$ derived from the new investment on Day 1 won't be available until Day 4. Therefore, we should delete the term $2x_1$ from the left-hand side of the third constraint and transfer this term to that of the fourth constraint. This results in revised constraints $s_2 = 0.5x_2 + x_3 + s_3$ and $2x_1 + s_3 = 0.5x_3 + x_4 + s_4$. After making a similar revision in the fifth constraint, we arrive at the following new formulation:

$$\begin{array}{ll} \text{Max} & 2x_4 + s_5 \\ \text{Subject to:} & \end{array}$$

$$\begin{aligned} s_0 &= x_1 + s_1 \\ s_1 &= 0.5x_1 + x_2 + s_2 \\ s_2 &= 0.5x_2 + x_3 + s_3 \\ 2x_1 + s_3 &= 0.5x_3 + x_4 + s_4 \\ 2x_2 + s_4 &= 0.5x_4 + s_5 \end{aligned}$$

$$x_j \geq 0 \text{ for } j = 1, 2, 3, 4 \text{ and } s_j \geq 0 \text{ for } j = 1, 2, 3, 4, 5.$$

Note that it is not necessary to revise the objective function.

4. Relax Assumption 3: Suppose the daily interest rate is 1%. Then, our (original) formulation revises to:

$$\begin{array}{ll} \text{Max} & 2x_4 + 1.01s_5 \\ \text{Subject to:} & \end{array}$$

$$\begin{aligned} s_0 &= x_1 + s_1 \\ 1.01s_1 &= 0.5x_1 + x_2 + s_2 \\ 2x_1 + 1.01s_2 &= 0.5x_2 + x_3 + s_3 \\ 2x_2 + 1.01s_3 &= 0.5x_3 + x_4 + s_4 \\ 2x_3 + 1.01s_4 &= 0.5x_4 + s_5 \end{aligned}$$

$$x_j \geq 0 \text{ for } j = 1, 2, 3, 4 \text{ and } s_j \geq 0 \text{ for } j = 1, 2, 3, 4, 5,$$

where every daily saving grows by a factor of 1.01 overnight.

5. Relax Assumption 4: When money is borrowed to make investments, we have a situation of leveraged investments. To formulate this situation, we introduce a new variable b_j, which is the amount borrowed for a day in the morning of day j before making investments on that day. The formulation then becomes

$$\begin{array}{ll} \text{Max} & 2x_4 + s_5 - 1.005b_5 \\ \text{Subject to:} & \end{array}$$

$$\begin{array}{rcll} s_0 = 100, \quad b_0 = 0, & & x_0 = 0, & \text{[Initial sunday position]} \\ s_0 + b_1 & = & 1.005b_0 + 0.5x_0 + x_1 + s_1, & \text{[Mon]} \\ 2x_0 + 1.01s_1 + b_2 & = & 1.005b_1 + 0.5x_1 + x_2 + s_2, & \text{[Tue]} \\ 2x_1 + 1.01s_2 + b_3 & = & 1.005b_2 + 0.5x_2 + x_3 + s_3, & \text{[Wed]} \\ 2x_2 + 1.01s_3 + b_4 & = & 1.005b_3 + 0.5x_3 + x_4 + s_4, & \text{[Thu]} \\ 2x_3 + 1.01s_4 + b_5 & = & 1.005b_4 + 0.5x_4 + \ 0 + s_5, & \text{[Fri]} \end{array}$$

$$x_j \geq 0 \text{ for } j = 1..4 \text{ and } s_j, b_j \geq 0 \text{ for } j = 1..5.$$

Here we pay 0.5% interest per day for borrowed money.

7 A Project Scheduling Formulation

Nathan and his roommates wake up late on Sundays and clean up their apartment as fast as possible to catch up with their friends for a ball game in the park. There are 8 cleaning steps. Some steps can be started only after some others are finished, the order among these steps is governed by **precedence relations**. For example, sink can only be cleaned after the dishes are washed. Predecessors and duration of each step is listed below:

Predecessor	Step	Duration (mins)
N/A	A = Clean the Fridge	10
N/A	B = Wash the Dishes	25
N/A	C = Make up the Beds	15
A , B	D = Clean the Sink	7
C , D	E = Take the Dust	18
E	F = Vacuum the Carpet	12
D	G = Take the Garbage out	3
F , G	H = Tidy up the Apartment	14

The question is how fast Nathan and his roommates can finish the cleaning while respecting the precedence relations. We make the following assumptions :

1. Nathan has plenty roommates (WLOG, say 8 people) to be assigned to the cleaning steps.
2. The number of people assigned to a step does not affect the duration of that step.

Thus, the work assignment is not the issue. We can focus on the timing of each step.

7.1 Decision Variables

We want to minimize the finishing time of the last step. Since H does not appear as a predecessor to any step, it is the last step. H is completed 14 minutes after its starting time (Assumptions 1 and 2). Thus, the starting times of the cleaning steps are sufficient to characterize the finishing times of all (including the last) step. Let

- t_j = Start time of step j, $j \in \{A, B, C, D, E, F, G, H\}$.

We assume that the first activity starts at time 0.

7.2 Objective Function

We want to minimize the finishing time of the last step:

$$\text{Min } t_H + 14.$$

7.3 Constraints

Because of the precedence relations, a step can only start after its predecessor is finished. We will use a single inequality constraint to represent each precedence relation. For example, "sink can only be cleaned after the dishes are washed" or B precedes D is represented as:

$$\text{Constraint B} \rightarrow \text{D} : t_D \geq t_B + 25$$

Similarly "sink can only be cleaned after the fridge is cleaned" or A precedes D is written as:

$$\text{Constraint A} \rightarrow \text{D} : t_D \geq t_A + 10$$

For each remaining precedence relation, we write a constraint:

$$\text{Constraint C} \rightarrow \text{E} : t_E \geq t_C + 15$$

$$\text{Constraint D} \rightarrow \text{E} : t_E \geq t_D + 7$$

$$\text{Constraint E} \rightarrow \text{F} : t_F \geq t_E + 18$$

$$\text{Constraint D} \rightarrow \text{G} : t_G \geq t_D + 7$$

$$\text{Constraint F} \rightarrow \text{H} : t_H \geq t_F + 12$$

$$\text{Constraint G} \rightarrow \text{H} : t_H \geq t_G + 3$$

On top of these 8 functional constraints, we add nonnegativity constraints: $t_A, t_B, t_C \geq 0$.

7.4 Remarks

1. Convince yourself that the nonnegativity constraints on D, E, F, G, H are not needed. These constraints are *implied* by functional constraints and the nonnegativity constraints on A, B and C.
2. Does the optimal solution change if we drop 14 from the objective function? How about the objective of Min $2t_H$, does the optimal solution change this time? Any generalizations?
3. Problems of this type are known as "activity scheduling" problems. Besides LP, a method called CPM (critical path method) can be used to obtain a solution.

8 Solved Exercises

1. Three US Olympic teams and their trainers will fly back from Sydney to San Fransisco with a plane that can carry 100 people. This will be a nonstop flight lasting 20 hours. Three teams are Swimming, Gymnastics and Cycling. These teams have the following number of members and trainers: Swimming 42 and 12; Gymnastics 22 and 14; Cycling 34 and 16. There must be at least one swimming trainer accompanying every three swimmers on the plane. Similarly, there must be at least one gymnastics trainer for every two gymnasts on the plane. Cyclists tend to be older and can travel by themselves without trainers. Swimming and cycling associations are equally paying for the trip and they first require that at least the 70% of the seats are allocated to swimmers, cyclists and their trainers. Second, the total number of swimmers and their trainers must equal to the total number of cyclist and their trainers.

a) Provide an LP formulation to minimize the number of people that cannot be put on this flight.

Let x_s, x_g, x_c, t_s, t_g, t_c be the number of swimmers, gymnasts, cycles, and their trainers put on the plane.

Min $\quad 140 - (x_s + x_g + x_c + t_s + t_g + t_c)$

ST :

$$x_s - 3t_s \leq 0$$
$$x_g - 2t_g \leq 0$$
$$x_s + t_s + x_c + t_c \geq 70$$
$$x_s + t_s - x_c - t_c = 0$$
$$0 \leq x_s \leq 42\ ,\ 0 \leq x_g \leq 22\ ,\ 0 \leq x_c \leq 34$$
$$0 \leq t_s \leq 12\ ,\ 0 \leq t_g \leq 14\ ,\ 0 \leq t_c \leq 16$$

b) What is the optimal value of the objective in a)? Justify your answer. You can answer this without solving the formulation.

Consider $x_s = 23$, $t_s = 12$, $x_g = 20$, $t_g = 10$, $x_c = 34$ and $t_c = 1$, this solution is feasible and yields an objective value of 40. You can pick another solution and discover that it also gives an objective value of 40 (consider $x_s = 24$, $t_s = 12$, $x_g = 18$, $t_g = 10$, $x_c = 30$ and $t_c = 6$). Indeed any feasible solution has an objective value of 40. Moreover, we can not reduce the objective value below 100, because the plain takes 100 people and we have 140 athletes.

c) Suppose that leaving out a gymnast costs three times as much as leaving out a swimmer or a cyclist. And also suppose that the cost of leaving out trainers is negligible. Modify your answer to a) to minimize the cost of people left behind (not put on the plane).

Modify the objective function as Min $(42 - x_s) + 3(22 - x_g) + (34 - x_c)$.

2. California has been experiencing electricity shortages. To solve the power shortage problem in the next 5 years, California government is going to invest \$1 B= \$1000 M into power plants. There are two types of power plants: Nuclear (N) and Hydroelectric (H). Every 1,000,000 dollars invested into N type plants results in n megawatts capacity in 2 years (time to complete an N type plant), the corresponding number is h in 3 years for H type plants. An operating plant requires an operating budget of 20% of its price. Operating budgets can not be used to start up new investments. But operating budgets or surplus money can be invested into money markets to accrue an interest at a rate of 10%. California currently has 50 megawatts of capacity and it can conserve energy to keep the demand about 50 megawatts in the next two years. However, it needs 66, 70 and 78 megawatts of additional power in the third, fourth and fifth years respectively. Assume that once a plant is operational, it is always operational.

a) Provide an LP formulation that meets the increasing power demand and maximizes the cash available to California at the end of the fifth year.

Let N_t and H_t be Nuclear and Hydroelectric investments (in million dollars) made in year t. Let I_t be the surplus money (including operating budgets) invested into money market in year t.

Max $1.1 I_5$

ST :

$$
\begin{aligned}
&1000 = I_1 + N_1 + H_1\\
&1.1 I_1 = I_2 + N_2 + H_2\\
&1.1 I_2 = I_3 + N_3\\
&1.1 I_3 = I_4\\
&1.1 I_4 = I_5\\
&n N_1 \geq 16\\
&n N_1 + n N_2 + h H_1 \geq 20\\
&n N_1 + n N_2 + h H_1 + n N_3 + h H_2 \geq 28\\
&I_3 \geq 0.2 N_1\\
&I_4 \geq 0.2 N_1 + 0.2(N_2 + H_1)\\
&I_5 \geq 0.2 N_1 + 0.2(N_2 + H_1) + 0.2(N_3 + H_2)\\
&N_t \geq 0\ ,\ H_t \geq 0,\ I_t \geq 0
\end{aligned}
$$

b) Suppose that environmentalist groups pressure California government so that the total Hydroelectric plant investment is required to be more than the total Nuclear plant investment. Modify your answer to (a).

Add the constraint: $N_1 + N_2 + N_3 \leq H_1 + H_2$.

c) Suppose that we have the inequality $n \leq h$, i.e., Hydroelectric plants are more efficient than Nuclear plants. Can we argue that there should be no Nuclear plant investment after the first year, why? If not, can you modify the inequality $n \leq h$ so that there is no Nuclear plant investment after the first year, how? By modification, we mean a new linear inequality that does not involve any decision variables or other parameters except n and h.

If $1.1n \leq h$, then we can reduce N_{t+1} by 1.1δ dollars and increase H_t by δ dollars. This operation does not harm power requirement constraints and still satisfies operating budget constraints. Moreover, we can still satisfy conservation of capital from t to $t+1$ with the interest rate. Note that I_3, I_4, I_5 values do not change with this operation. Consequently any optimal solution that has $N_{t+1} = \delta$ and $H_t = \epsilon$ leads to another optimal solution where $N_{t+1} = 0$ and $H_t = \epsilon + \delta$. When $n \leq h \leq 1.1n$, this argument breaks. This is because capital can not be conserved from t to $t+1$ while power requirements are satisfied.

3. PlanoTurkey produces two types of turkey cutlets for sale to fast food restaurants. Each type of cutlet consists of white meat and dark meat. Cutlet 1 sells for \$8/kg and must consist of at least 70% white meat. Cutlet 2 sells for \$6/kg and must consist of at least 60% white meat. At most 50 kg of cutlet 1 and 30 kg of cutlet 2 can be sold for Thanksgiving. Two types of turkey used to manufacture the cutlets are purchased from an Addison farm. Each type 1 turkey costs \$10 and yields 5 kg of white meat and 2 kg of dark meat. Each type 2 turkey costs \$8 and yields 3 kg of white meat and 3 kg of dark meat. Formulate a LP to maximize PlanoTurkey's profit.

a) Define decision variables.

T_1: Number of type 1 turkey purchased. D_1: Kilograms of dark meat used in cutlet 1. W_1: Kilograms of white meat used in cutlet 1. Define T_2, D_2, W_2 similarly.

b) Profit is revenue minus costs, express the profit in terms of decision variables.

Max $8(W_1 + D_1) + 6(W_2 + D_2) - 10T_1 - 10T_2$

c) Write constraints so that no more cutlets than demand is sold.

$W_1 + D_1 \leq 50$ and $W_2 + D_2 \leq 30$

d) Write constraints so that PlanoTurkey does not use more white or dark meat than it buys from the Addison farm.

$W_1 + W_2 \leq 5T_1 + 3T_2$ and $D_1 + D_2 \leq 2T_1 + 3T_2$

e) Finish your formulation by adding any contraints you find necessary.

Cutlet 1 must have at leat 70% white meat:

$$\frac{W_1}{W_1 + D_1} \geq 0.7.$$

Cutlet 2 must have at leat 60% white meat:

$$\frac{W_1}{W_1 + D_1} \geq 0.6.$$

Nonnegativity constraints: $T_1, D_1, W_1, T_2, D_2, W_2 \geq 0$

4. The Apex Television company has to decide on the number of 27 and 20 inch sets to be produced at one of its factories. Market research indicates that at most 40 of the 27 inch sets and 10 of the 20 inch sets can be sold per month. The maximum number of work hours available is 800 hours per month. A 27 inch set requires 15 work hours and a 20 inch set requires 10 work hours . Each 27 inch set produces a profit of \$120 and the same number is \$80 for 20 inch sets.
a) Formulate an LP to maximize the profit:

B: Number of 27 inch sets produced per month. S: Number of 20 inch sets produced per month.

Max $120B + 80T$
Subject to:
$B \leq 40$
$S \leq 10$
$15B + 10S \leq 800$
$B, S \geq 0$

b) Through commercials, TV set demand can be increased. For every \$20 spent for commercials, 1 more 27 inch TV **and** 2 more 20 inch TV can be sold. Formulate an LP to maximize the profit with a budget of \$400 for commercials.

C: Commercial budget spent for TVs.

Max $120B + 80T - C$
Subject to:
$B \leq 40 + C/20$
$S \leq 10 + C/10$
$C \leq 400$
$15B + 10S \leq 800$
$B, S \geq 0$

5. Sweet Co. produces two types of candies; energy candy and happy candy. Ingredients for both candies are sugar, nuts and chocolate. Currently, the company has 100 kg of sugar, 20 kg of nuts and 30 kg of chocolate in stock. The mixture of energy candy contains at least 20% nuts. The mixture of happy candy contains at least 10% nuts and 15% chocolate. Assume that one kg of each ingredient contributes one kg to the final candy product. Each kg of energy candy can be sold at \$25 and each kg of happy candy can be sold at \$20. State your decision variables and formulate an LP to maximize the revenue.

Let E, H be the kgs of candy produced. Let ES, EN, EC be the Sugar, Nut, Chocolate used in Energy candy, similarly define HS, HN, HC.
Max $25E + 20H$
ST:
$E = ES + EN + EC$
$H = HS + HN + HC$
$ES + HS \leq 100$

$EN + HN \leq 20$
$EC + HC \leq 30$
$EN \geq 0.20E$
$HN \geq 0.10H$
$HC \geq 0.15H$
All variables nonegative.

6. Consider the warehouse space requirement problem for a web mercantile for the next five months. The mercantile leases space at the following costs:

Lease Duration (Months)	1	2	3	4	5
Cost ($\$/m^2$)	650	1000	1350	1600	1900

The payments for the space are made in full in the month the leasing contract is signed. Let $x_{i,j}$ be space leased in month i for j months. Consider the LP below:
max $\quad x_{11} + 2x_{12} + 3x_{13} + 4x_{14} + 5x_{15} + x_{21} + 2x_{22} + 3x_{23} + 4x_{24} + x_{31} + 2x_{32} + 3x_{33} + x_{41} + 2x_{42} + x_{51}$
ST:
$650x_{51} \leq 50000$
$650x_{31} + 1000x_{32} + 1350x_{33} \leq 50000$
$650x_{11} + 1000x_{12} + 1350x_{13} + 1600x_{14} + 1900x_{15} \leq 65000$
$650x_{21} + 1000x_{22} + 1350x_{23} + 1600x_{24} \leq 55000$
$650x_{41} + 1000x_{42} \leq 40000$
All $x_{ij} \geq 0$

a) Express the objective function in words. Suppose that constraints are written to make sure that leasing costs are whitin monthly budgets, fill in the table below:

Objective maximizes the total space rented in 5 months.

Months	1	2	3	4	5
Budget ($K)	65	55	50	40	50

b) Provide an LP formulation to maximize the minimum space available for storage in 5 months.

Let z be the minimum space available in 5 months then

$$z = \min \left\{ \begin{array}{ll} x_{11} + x_{12} + x_{13} + x_{14} + x_{15} & \text{for month 1} \\ x_{12} + x_{13} + x_{14} + x_{15} + x_{21} + x_{22} + x_{23} + x_{24} + x_{25} & \text{for month 2} \\ x_{13} + x_{14} + x_{15} + x_{22} + x_{23} + x_{24} + x_{31} + x_{32} + x_{33} & \text{for month 3} \\ x_{14} + x_{15} + x_{23} + x_{24} + x_{32} + x_{33} + x_{4,1} + x_{4,2} & \text{for month 4} \\ x_{1,5} + x_{2,4} + x_{3,3} + x_{4,2} + x_{5,1} & \text{for month 5} \end{array} \right\}.$$

Replace the objective with max $\quad z$ and insert the following constraints:
$z \leq x_{11} + x_{12} + x_{13} + x_{14} + x_{15}$
$z \leq x_{12} + x_{13} + x_{14} + x_{15} + x_{21} + x_{22} + x_{23} + x_{24} + x_{25}$
$z \leq x_{13} + x_{14} + x_{15} + x_{22} + x_{23} + x_{24} + x_{31} + x_{32} + x_{33}$
$z \leq x_{14} + x_{15} + x_{23} + x_{24} + x_{32} + x_{33} + x_{4,1} + x_{4,2}$
$z \leq x_{1,5} + x_{2,4} + x_{3,3} + x_{4,2} + x_{5,1}$

7. A company produces two products (1,2) using two machines (M,N). Product 1 requires processes on **both** machines M and N. On the contrary product 2 can be produced on **either** machine M or N. Processing times (in minutes) on each machine are

Product	Machine N	Machine M
1	15	18
2	20	25

Each machine works for 8 hours every day. Due to marketing limitations the number of Product 1 sold must be at least the number of Product 2 sold. When sold, each unit of Product 1 and 2 contributes to profit \$16 and \$20.

a) Provide an LP to maximize daily contribution to profit.

Let x_1 be the number of Product 1 produced, and x_{2M} and x_{2N} be the number of Product 1 and 2 produced on machines M and N.
Max $16x_1 + 20(x_{2M} + x_{2N})$
ST
$15x_1 + 20x_{2N} \leq 8(60)$
$18x_1 + 25x_{2N} \leq 8(60)$
$x_1 \geq x_{2M} + x_{2N}$
$x_1, x_{2M}, x_{2N} \geq 0$

b) Suppose that marketing limitation is lifted. Then compute how many more Product 2 can be produced by producing one fewer Product 1, if the capacity usage is not altered. In this case, compare the reduction in profit due to Product 1 against the increase in profit due to Product 2. Finally argue that Product 1 will not be produced in the optimal solution.

With one less Product 1, 15 mins and 18 mins capacity are released on Machines N and M. This capacity can be used to produce 15/20 and 18/25 Product 2 on machines N and M. The net effect to profit is -16+(15/20)20+(18/25)20 and is positive. Reducing Product 1 production increases profit so no Product 1 is produced in the optimal solution.

8. An emergency room operates for 24 hours and have the following number of cases during the indicated time slots:

Times	08-12	12-16	16-20	20-24	24-04	04-08
Cases/hr	2	11	7	10	3	2

There are three shifts starting at 8:00, 16:00 and 24:00. All nurses that start at these times can handle 2 cases per hour and work 8 consecutive hours. Some nurses ask to start at 12:00 or 20:00 and claim that they can work more productively handling 3 cases per hour if they let to start at 12:00 or 20:00 and work 8 consecutive hours. The emergency room pays equal salary to all nurses except for those working 24:00-08:00 shift, who earn 20% more.
a) Formulate an LP to minimize total salaries paid.

Let x_1 be the number of nurses starting at 08:00, x_2 starting at 12:00, x_3 starting at 16:00, x_4 starting at 20:00, x_5 starting at 24:00. Nobody starts at 04:00.
Min $x_1 + x_2 + x_3 + x_4 + 1.2x_5$
$2x_1 \geq 2$
$2x_1 + 3x_2 \geq 11$
$3x_2 + 2x_3 \geq 7$
$2x_3 + 3x_4 \geq 10$
$3x_4 + 2x_5 \geq 3$

$2x_5 \geq 2$
$x_1, x_2, x_3, x_4, x_5 \geq 0$

b) Argue that we can find an optimal solution where exactly 1 nurse starts at 08:00.

If two nurses were to start at 08:00, we can shift one nurse to 12:00-20:00 slot. Doing so we **do not increase** the objective value. Moreover first two constraints will still be satisfied. Hence, we can decrease the number of nurses starting at 8:00 down to 1 and find an optimal solution. Then we can set $x_1 = 1$.

c) Suppose that only one nurse starts at 08:00 and argue that no nurse starts at 16:00 in an optimal solution.

If one nurse starts at 08:00, 3 nurses must start at 12:00 to cover 11 cases per hour. Three nurses can continue to cover 7 cases per hour during 16:00-20:00. No new nurses need to start at 16:00. We can set $x_3 = 0$.

9. Farmer Billy Bauer has two farms in Dallas to grow wheat and barley. There are differences in the yields and costs of growing crops due to soil conditions at two farms:

	McKinney Farm	Addison Farm
Barley yield/acre	400 bushels	700 bushels
Cost/acre of barley	\$90	\$80
Wheat yield/acre	350 bushels	300 bushels
Cost/acre of wheat	\$110	\$100

McKinney and Addison farms have 70 and 120 acres for cultivation. At least 20000 bushels of barley and 30000 bushels of wheat must be grown. Provide an LP to minimize the cost of meeting wheat and barley demand.

Let BM: Area in acres dedicated for Barley production at McKinney. BA: Area in acres dedicated for Barley production at Addison. Similarly define WM and WA.

Min $90BM + 80BA + 110WM + 100WA$
ST:
$BM + WM \leq 70$
$BA + WA \leq 110$
$400BM + 700BA \geq 20000$
$350WM + 300WA \geq 30000$
$BM, BA, WM, WA \geq 0.$

10. The CapMan Capital managing group is going to invest \$10,000,000 of a large pension fund. The CapMan has identified six mutual funds with varying investment strategies, resulting in different potential returns:

Pension fund	1	2	3	4	5	6
Price (\$/share)	50	75	80	30	40	20
Expected return %	30	25	20	15	10	5
Risk category	High	High	High	Medium	Medium	Low

CapMan reduces the risk of its portfolio making sure that

1. The total amount invested in high risk funds must be between 50 and 75% of the total portfolio.

2. The total amount invested in low risk funds must be at least 5% of the total portfolio.

a) Provide an LP formulation to maximize the expected total return from the funds.

Let x_i be the investment made into fund i.

Max $0.30x_1 + 0.25x_2 + 0.20x_3 + 0.15x_4 + 0.10x_5 + 0.05x_6$
ST:
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10,000,000$
$x_1 + x_2 + x_3 \leq 7,500,000$
$x_1 + x_2 + x_3 \geq 5,000,000$
$x_6 \geq 500,000$
$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Share prices do not play a role in deciding how much to invest into funds. But if you want to find out how many shares to buy, just divide the investments by the share prices.

b) CapMan is considering to manage the risk by diversifying the investments. It requires that the investment into funds 1 and 2 are equal and also that the the investment into fund 4 is twice the investment into fund 5. Modify your formulation in (a) to reflect this new policy.

Add the constraints: $x_1 = x_2$ and $x_4 = 2x_5$.

c) CapMan president is wondering if the diversification policy of (b) would pull the returns of (a) down. Using LP terminology explain what would happen to the returns.

Returns will perhaps go down because constraints in (b) might cut out the optimal solution.

11. Consider the warehouse space requirement problem for a web mercantile for the next five months. The mercantile leases space at the following costs:

Leasing Period (Months)	1	2	3	4	5
Cost ($\$/m^2$)	650	1000	1350	1600	1900

The mercantile has a monthly budget to make payments for the leases:

Months	1	2	3	4	5
Budget ($K)	65	55	50	40	50

The payments for the space are made in full in the month the leasing contract is signed.
a) Provide an LP to maximize the sum, over all months, the total area available in each month. For example, if we lease $100m^2$ in the first month for two months, this space is available for two months, it contributes 2x$100m^2$ to the objective.

$x_{i,j}$: Space leased in month i for j months.

$Max\ \ x_{11} + 2x_{12} + 3x_{13} + 4x_{14} + 5x_{15} + x_{21} + 2x_{22} + 3x_{23} + 4x_{24} + x_{31} + 2x_{32} + 3x_{33} + x_{41} + 2x_{42} + x_{51}$
ST:
$650x_{11} + 1000x_{12} + 1350x_{13} + 1600x_{14} + 1900x_{15} \leq 65000$
$650x_{21} + 1000x_{22} + 1350x_{23} + 1600x_{24} \leq 55000$

$650x_{31} + 1000x_{32} + 1350x_{33} \leq 50000$
$650x_{41} + 1000x_{42} \leq 40000$
$650x_{51} \leq 50000$
All $x_{ij} \geq 0$

b) Looking at your formulation in (a), can you argue that leasing space for one month in the first month cannot be optimal.

Indeed, $x_{11} = 0$ in the optimal solution. If that is not the case, we reduce x_{11} by 1 unit and increase x_{12} by 650/1000 unit. the net effect of this change on the objective function is $-1 + 2(650/1000) > 0$, so such a change improves the solution. Interestingly, you can similarly argue that $x_{12} = x_{13} = x_{14} = 0$.

You can obtain the values of the decision varaiables by inspection. You can split the above formulation into five, one formulation per month:

For the first month:
$Max \quad x_{11} + 2x_{12} + 3x_{13} + 4x_{14} + 5x_{15}$
ST:
$650x_{11} + 1000x_{12} + 1350x_{13} + 1600x_{14} + 1900x_{15} \leq 65000$
All $x_{ij} \geq 0$
The optimal solution is $x_{11} = x_{12} = x_{13} = x_{14} = 0$ and $x_{15} = 65000/1900$.

For the second month:
$Max \quad x_{21} + 2x_{22} + 3x_{23} + 4x_{24}$
ST:
$650x_{21} + 1000x_{22} + 1350x_{23} + 1600x_{24} \leq 55000$
All $x_{ij} \geq 0$
The optimal solution is $x_{21} = x_{22} = x_{23} = 0$ and $x_{24} = 55000/1600$.

For the third month:
$Max \quad x_{31} + 2x_{32} + 3x_{33}$
ST:
$650x_{31} + 1000x_{32} + 1350x_{33} \leq 50000$
All $x_{ij} \geq 0$
The optimal solution is $x_{31} = x_{32} = 0$ and $x_{33} = 50000/1350$.

Repeating for the fourth and the fifth month: $x_{4,1} = 0$, $x_{4,2} = 40000/1350$ and $x_{51} = 50000/650$. LPs that allow for such separation are called separable Linear programs.

12. Two cities generate waste and their wastes are sent to incinerators (=furnaces) for burning. Daily waste production and distances among cities and incinerators are below:

	Waste produced tons/day	Distance to A in miles	incinerator B in miles
City 1	500	30	20
City 2	400	36	42

Incineration reduces each ton of waste to 0.2 tons of debris, which must be dumped at one of the two landfills. It costs \$3 per mile to transport a ton of material (either debris or waste). Distances (in miles) among incinerators and landfills are shown below:

	Capacity tons/day	Incineration cost dollars/ton	Distance to Northern in miles	landfills Southern in miles
Incinerator A	500	40	5	8
Incinerator B	600	30	9	6

Incineration capacity and cost is based on the amount of waste input.
a) Formulate an LP that can be used to minimize the total cost of disposing waste of both cities.

Decision variables: Let $x_{i,j}$ be the waste sent from city i to incinerator j, $i \in \{1, 2\}$ and $j \in \{A, B\}$. Let $y_{j,k}$ be the waste sent from incinerator i to ladfill j, $j \in \{A, B\}$ and $k \in \{S, N\}$. Some of these are consequential

Objective function: Minimize $3(30x_{1A}+20x_{1B}+36x_{2A}+42x_{2B}+5y_{AN}+8y_{AS}+9y_{BN}+6y_{BS})+40(x_{1A}+x_{2A}) + 30(x_{1B} + x_{2B})$

Constraints:
Waste produced per day: $500 = x_{1A} + x_{1B}$, $400 = x_{2A} + x_{2B}$.
Incinerator capacities: $x_{1A} + x_{2A} \leq 500$, $x_{1B} + x_{2B} \leq 600$.
Waste reduction: $0.2(x_{1A} + x_{2A}) = y_{AN} + y_{AS}$, $0.2(x_{1B} + x_{2B}) = y_{BN} + y_{BS}$.
Nonnegativity constraints.
b) After a quick look at the distance between cities and incinerators and also between incinerators and landfills, we conclude if incinerators are moved into the cities the problem data will change as:

	Waste produced tons/day	Distance to A	incinerator B
City 1	500	0	30
City 2	400	30	0

	Capacity tons/day	Incineration cost dollars/ton	Distance to Northern	landfills Southern
Incinerator A	500	40	35	38
Incinerator B	600	30	51	48

Explain why such moves may reduce transportaion costs. Now modify your LP to minimize total costs.

These moves save transportation costs because after burning only a small portion of waste remains for transportation. Only the objective function needs modification.
Objective function: Minimize $3(0x_{1A}+30x_{1B}+30x_{2A}+0x_{2B}+35y_{AN}+38y_{AS}+51y_{BN}+48y_{BS})+40(x_{1A}+x_{2A}) + 30(x_{1B} + x_{2B})$

c) By moving the incinerators into cities, the solution in b) will improve the optimal objective value of the formulation in a). Would you approve the move or is there anything the analysis is missing, why? Answer in at most 4 sentences.

The move cannot be approved. It can cause tremendous health problems for city population if the incinerators release the air without filtering. If additional filters are to be built, they cost extra. Neither health nor extra filter costs is captured in our formulations.

The answers to this part can be very creative. What is to remember is that the models capture only some aspects of the real life so before implementing their solution, we must take a look at the "bigger picture".

13. At a college of engineering the dean is planning for the full time professor needs for the next 4 years. After studying the projected total credit hours offered in each year, she finds out the required number of professors to be as follows:

Year	2005	2006	2007	2008
# of Profs required	72	94	69	103

From her past experience, she knows that professors prefer long term job contracts and that they would accept a lower salary if the contract is longer:

Contract length in years	1	2	3	4
Annual salary (in K dollars)	90	80	70	60

For example, if a professor is needed for two years, it is better to sign a two year contract and pay \$160 K over the duration of the appointment. Compare this number against \$180 K which must be paid if two 1-year contracts are signed. Suppose that there currently are no contracts with any professor. Formulate a linear program to minimize the total salaries paid to professors at this school while ensuring that there are enough professors signed up for each year. Please define your decision variables as exact as possible.

x_{ij}: Number of professors signed in year i for j years. Since we are looking at only 4 years, the decision variables are defined only for $4+1 \geq i+j$.
We will use the total salaries paid over a contract to a single professor in our objective function:

Min $90(x_{1,1}+x_{2,1}+x_{3,1}+x_{4,1})+160(x_{1,2}+x_{2,2}+x_{3,2})+210(x_{1,3}+x_{2,3})+240x_{1,4}$
Subject to:
$x_{1,1}+x_{1,2}+x_{1,3}+x_{1,4} \geq 72$
$x_{1,2}+x_{1,3}+x_{1,4}+x_{2,1}+x_{2,2}+x_{2,3} \geq 94$
$x_{1,3}+x_{1,4}+x_{2,2}+x_{2,3}+x_{3,1}+x_{3,2} \geq 69$
$x_{1,4}+x_{2,3}+x_{3,2}+x_{4,1} \geq 103$
$x_{i,j} \geq 0$

14. The Flora pharmaceutical of New Jersey produces a weight loss drug called Manizac. Manizac is made of 3 chemicals and requires two active ingredients, called A and B, found in these chemicals. The cost per kg of each chemical and the amount of each active ingredient (by weight) in each chemical is given as

Chemical	cost	A	B
1	14	0.05	0.06
2	13	0.06	0.04
3	10	0.11	0.02

The FDA requires that Manizac must consist of, by weight, at least 8% of ingredient A and 5% of ingredient B. Assume that the weight is conserved during the drug production.
a) Formulate a linear program for Flora that will minimize raw material costs to produce 1 kg of Manizac.

Let x_1, x_2 and x_3 be the amount (in kg) of chemicals 1,2 and 3 used in the production.

Min $14x_1+13x_2+10x_3$
Subject to:
$x_1+x_2+x_3=1 \quad (1)$
$0.05x_1+0.06x_2+0.11x_3 \geq 0.08(x_1+x_2+x_3)=0.08 \quad (2)$
$0.06x_1+0.04x_2+0.02x_3 \geq 0.05(x_1+x_2+x_3)=0.05 \quad (3)$

$x_1, x_2, x_3 \geq 0$

b) Now suppose that Flora wants to produce 16 kg of Manizac. Redefine your decision variables in a) appropriately so that an optimal solution to your formulation in a) can be used to find an optimal solution for producing 16 kg of Manizac. For example, if we know that the optimal solution to a) involves using 0.25 kg of Chemical 1 and 0.75 kg of Chemical 3, what would be the optimal solution for producing 16 kg of Manizac?

Let x_1, x_2 and x_3 be the amount (in 16 kgs) of chemicals 1,2 and 3 used in the production. Then the formulation of a) does not change. We have just changed the units of measurement for the decision variable. For example, the optimal solutions is still $x_1 = 0.25$ and $x_3 = 0.75$, however these numbers now correspond to 4 kg of Chemical 1 and 12 kg of Chemical 2.

c) Write the name of the chemical that will not be used in an optimal solution. Justify your answer.

Suppose that the optiaml solution is (x_1^*, x_2^*, x_3^*) where $x_2^* > 0$. We claim that $(x_1^* + 0.5x_2^*, 0, x_3^* + 0.5x_2^*)$ is a better solution than (x_1^*, x_2^*, x_3^*). Since an equal combination of chemicals 1 and 3 costs 12 and is chepaer than chemical 2, the objective function will be lower with the solution $(x_1^* + 0.5x_2^*, 0, x_3^* + 0.5x_2^*)$. We are left to argue that the solution $(x_1^* + 0.5x_2^*, 0, x_3^* + 0.5x_2^*)$ is feasible.

The first constraint continues to be satisfied with the new solution because it is satisfied with the old solution and the sum of these two solutions are the same and equal to 1. The second constraint makes sure that there is enough A in Manizac. An equal combination of chemical 1 and 3 brings in (0.05+0.11)/2=0.08 kg of A while the same amount of chemical 2 brings in only 0.06 kg of A. On the other hand, an equal combination of chemical 1 and 3 brings in (0.06+0.02)/2=0.04 kg of A while the same amount of chemical 2 brings also 0.04 kg of A. Thus, an equal combination of chemical 1 and 3 dominate chemical 2 in terms of active ingredient A and B content. As a result, constraints (2) and (3) continue to hold with the new solution: the new solution is also feasible.

9 Exercises

1. A furniture manufacturer produces and sells TV stands at a price of \$100. If 10 or fewer stands are manufactured per week, each stand costs \$60. Manufacturer's regular capacity is 10 stands per week. Manufacturer can hire additional workforce to bring its capacity up to 20 stands per week. However, additional capacity is costly so any stand produced after the 10th costs \$75. For example 12 stands cost $10 \cdot 60 + 2 \cdot 75 = 750$ dollars. Manufacturer has a weekly operating capital of \$1200 so its weekly costs can not exceed this amount. It also has committed to deliver to a customer 2 stands every week. Provide an LP formulation to maximize the manufacturer's profit.

2. A young college professor Dr.Martin decides to supplement his income by raising chicken and rabbits in his balcony. Each rabbit sells at \$25 at the farmer's market and the price for a chicken is \$15. Rabbits (chickens) are sold 20 (18) weeks after their birth. Dr.Martin has a 12 m^2 balcony and each rabbit (chicken) needs 2 (0.5) m^2 living area. Raising a rabbit (chicken) costs \$1 (\$0.6) (these are mainly food costs) and Dr.Martin has only \$5 per week to spend on rabbit/chicken food.
   a) Provide an LP formulation to maximize Dr.Martin's revenue from rabbit and chicken sales.
   b) Dr.Martin realizes that his chickens are too aggressive and are attacking young rabbits. He decides to raise at least four times as many rabbits as chickens so that rabbits can defend themselves. Modify the formulation to reflect this restriction.

3. Because of the poor air quality in Dallas (partly due to too many private cars in the traffic), DART wants to start up 4 new routes: between Plano and Dallas, between Richardson and Addison, between Plano and Richardson, and between Richardson and Dallas. Plano and Dallas municipalities put down \$800,000 and \$200,000, respectively, to finance these routes. DART is facing the question of investing a total of \$1,000,000 (=800,000+200,000) to four routes such that at least 80% of the investment is made for the routes involving Plano and at least 30% of the investment is made for the routes involving Dallas. Richardson - Addison and Plano - Richardson routes are almost of equal length and DART can make a profit at the rate of 20% per year for each dollar invested. These rates are 0% for the Plano - Dallas route and 10% for the Richardson - Dallas route.
   a) Give an LP formulation that maximizes DART's return on investment in a year.
   b) Argue that there can be an optimal solution without investing into Richardson - Addison route.

4. Suppose that x_i denotes a decision variable. Which of the following mathematical relationships can be found among the constraints of an LP whose decision variables are x_1, x_2, x_3, x_4:

   (a) $x_1 - 2x_2 + 3x_4 \geq -8$
   (b) $3x_1 - x_2x_3 = 3$
   (c) $(x_2 + x_3)/x_1 \leq -7$
   (d) $x_1^2 + x_2^2 - x_3^2 = 0$
   (e) $x_1^{x_2} = 2$
   (f) $\sin(\pi/2)x_1 + 10^8 x_2 = 1$
   (g) $x_2^3 = 8$

   Basically, identify the relationships which are linear inequalities or linear equalities.
   For each relationship that is not a linear inequality, see if you can convert it to one or more linear inequalities or equalities. For example, some inequalities that involve max on the right-hand side can be converted into two linear inequalities. But there are other ways of converting relationships into linear inequalities; recall those. When you are converting/reducing these relationships into linear inequalities or equalities, you must come up with linear inequalities or equalities which are equivalent to the original relationship in the following sense. Both the original relationship and your linear inequalities or equalities must have exactly the same solution(s).
   While attempting to convert a nonlinear function into a linear function, you should **not** change variables. Let us illustrate with an example what you should **not** do. Clearly $f(x_1, x_2) = x_1^2 - x_2$ is a nonlinear function of x_1. If you set $y := x_1^2$ to obtain $f(\sqrt{y}, x_2) = y - x_2$ and define $g(y, x_2) := f(\sqrt{y}, x_2)$. Then g is a linear function of y and x_2. But the question is whether f itself is linear in x_1 and x_2. The *equivalent* linear inequalities that you provide must have only x_1, x_2, x_3, x_4 as decision variables.

5. Textbook H-L: p. 95 3.4.9)a., pp. 96-97 3.4.13)a. and b.

6. Textbook H-L: p.98-99 3.4-19)a. (Do not do part b).

7. Solve H-L p.98, 3.4.17 a)

8. Cheese Spoilage: Refer to Production Planning Example. Suppose we are modelling cheese production and cheese can be kept at most a month in storage before consumption. If it is produced in Jan, it can be kept in the inventory during Feb and can be used to meet Feb demand. But that cheese cannot be used to meet Mar demand. We also suppose that 3 Assumptions made for production planning example are still valid. Modify the production planning LP to model cheese production.
   Some of you may think that in this case the inventory at the end of month j must always be $x_j - d_j$.

We illustrate that this is not necessarily so with the following example. Say $d_1 = 30$,$d_2 = 50$, $x_1 = 50$ and $x_2 = 40$. Starting with $y_0 = 0$, $y_1 = 20$. In Feb, we use y_1 to meet 20 of 50 units of demand and use x_2 to meet remaining 30 of 50 units of demand. Eventually, $y_2 = 10$ units of x_2 is passed to Mar as inventory. But

$$10 = y_2 \neq x_2 - d_2 = -10 \qquad \text{while} \qquad 10 = y_2 = x_1 - d_1 + x_2 - d_2 = 20 + (-10).$$

Thus, saying $y_j = x_j - d_j$ is an incorrect answer.

9. Refer to the Cheese Spoilage problem. Lee, a Cohort MBA student, defined $p_1 := 0$ and $p_j = \max\{y_{j-1} - d_j, 0\}$ for $j \geq 2$. Then he modified the inventory balance equations to have

$$y_j = y_{j-1} + x_j - d_j - p_j \quad \text{for } j \geq 1.$$

a) This student forgot to define his new variable p_j. To complete his solution, express what p_j is in English.
b) By plugging $p_j = \max\{y_{j-1} - d_j, 0\}$ into Lee's inventory balance equation, it is possible to mold that equation into

$$y_j = \min\{y_{j-1} - d_j, 0\} + x_j \quad \text{for } j \geq 1.$$

However, the min above cannot be included in a linear program. To express this min, supose we let $\bar{y}_j := \min\{y_{j-1} - d_j, 0\}$, introduce inequalities $\bar{y}_j \leq y_{j-1} - d_j$ and $\bar{y}_j \leq 0$, and have $y_j = \bar{y}_j + x_j$. To complete this *equivalent* representation, we must argue that $\bar{y}_j = y_{j-1} - d_j$ or $\bar{y}_j = 0$ in an optimal solution. That is we must argue that we cannot have $\bar{y}_j < y_{j-1} - d_j$ or $\bar{y}_j < 0$ in an optimal solution. Can you make such an argument? Note: This is an open question.
c) Instead of trying to linearize $\min\{y_{j-1} - d_j, 0\}$ as in b), suppose we claim that $y_{j-1} \leq d_j$ in an optimal solution. Make an argument for this claim and use the claim to add $y_{j-1} \leq d_j$ to the original inventory balance equations to complete the formulation.

10. TexOil Company can buy two types of crude oil: light oil at \$20/barrel and heavy oil \$17/barrel. When a barrel of oil is refined it yields gasoline, jet fuel and kerosene in the following quantities (in terms of barrel of output per barrel of input):

	Gasoline	Jet Fuel	Kerosene
Light oil	0.43	0.20	0.28
Heavy oil	0.32	0.38	0.21

TexOil has promised to deliver 800,000 barrels of gasoline, 1,000,000 barrels of Jet Fuel and 300,000 barrels of Kerosene. Provide an LP formulation that keeps promises and minimizes total oil purchase cost.

11. PlaCar company manufactures light truck and SUV bodies. The production requires certain amount of steel and labor as shown below with the availability and cost information:

	Steel (kgs)	Labor (hrs)
Truck body	1480	18
SUV body	1820	15
Unit cost (\$)	3	12
Total available	1,000,000	15000

According to forecasts at most 800 Truck bodies at \$6000 each and 650 SUV bodies at \$7200 each can be sold.

a) Provide an LP to maximize PlaCar's profit.

b) Big vehicles consume too much gas, suppose that EPA (Environmental Protection Agency) charges a regulatory fine to body manufacturers of \$500 for each body that weighs more than 1500 kgs. How would you modify your formulation in a).

c) Light truck bodies are narrower than SUV bodies. To bring down the required amount of steel for SUVs down to 1480kgs and to avoid EPA's fine, PlaCar decides to use a slight modification of light truck body for its SUVs. How would you modify your formulation in a).

d) Given the formulation in c), can we deduce that PlaCar will produce 650 SUVs and several trucks, explain.

12. Loan Portfolio Model: The Bank of America is formulating a loan policy to utilize a maximum of \$100 millionofits assets. The following table provides data for different types of loans: Unrecoverable

Loan	Interest	Chance of no recovery
Farm	0.05	0.01
Home	0.08	0.03
Car	0.12	0.05
Grad Education	0.14	0.08
Venture capital	0.15	0.20

loans do not generate interest income. Bank of America wants to dedicate at least 30% of its loans for education but the chance of no recovery for (weighted average by loan sizes of) the loan portfolio should not exceed 0.06. Formulate an LP to maximize interest income.

13. A Marketing Model: A phone survey company needs to survey at least 100 wives, 90 husbands, 80 single adult males and 70 single adult females. Making a daytime phone call costs 1.5 dollars per call, the same number is 3 for nighttime calls. The likelihood of the type of the person who would pick up the phone during the day and the night are different but given by the following table:

	Wife	Husband	Single adult male	Single adult female	Total
Day time	0.25	0.15	0.3	0.3	1.0
Night time	0.25	0.35	0.15	0.25	1.0

Provide an LP formulation to complete this survey at minimum cost.

14. Investment with Limited Leverage: Recall the investment formulation with borrowing

$$
\begin{array}{ll}
\text{Max} & 2x_4 + s_5 - 1.005b_5 \\
\text{Subject to:} &
\end{array}
$$

$$
\begin{array}{rcll}
s_0 = 100, \quad b_0 = 0, & & x_0 = 0, & \text{[Initial sunday position]} \\
s_0 + b_1 & = & 1.005b_0 + 0.5x_0 + x_1 + s_1, & \text{[Mon]} \\
2x_0 + 1.01s_1 + b_2 & = & 1.005b_1 + 0.5x_1 + x_2 + s_2, & \text{[Tue]} \\
2x_1 + 1.01s_2 + b_3 & = & 1.005b_2 + 0.5x_2 + x_3 + s_3, & \text{[Wed]} \\
2x_2 + 1.01s_3 + b_4 & = & 1.005b_3 + 0.5x_3 + x_4 + s_4, & \text{[Thu]} \\
2x_3 + 1.01s_4 + b_5 & = & 1.005b_4 + 0.5x_4 + \ 0 + s_5, & \text{[Fri]}
\end{array}
$$

$x_j \geq 0$ for $j = 1..4$ and $s_j, b_j \geq 0$ for $j = 1..5$.

Suppose that the debt-to-equity ratio is restricted by a financial regulation to be 100%, so add the following limited-leverage constraints to the formulation:

$$b_1 \le s_0, \ b_2 \le s_1, \ b_3 \le s_2, \ b_4 \le s_3, \ b_5 \le s_4.$$

a) With the limited-leverage constraints, we have an investment formulation where debt cannot exceed equity. Solve this formulation by using Excel Solver. Report the objective function value and the optimal values of all of the decision variables.
b) Identify the limited-leverage constraints that are satisfied as equality in the optimal solution of part a).
c) Increase the debt-to-equity ratio to 200% to mimic an unregulated financial market. Solve the formulation again in Excel Solver and report only the objective value. Comment on the reason for the difference between the objective values computed in a) and in c).

15. Alternative Formulation for Investment with Limited Leverage: Consider the alternative formulation for investment with borrowing

$$\begin{array}{ll}
\text{Max} & 2x_4 + s_5 - b_5 - 0.005(b_1 + b_2 + b_3 + b_4 + b_5) \\
\text{Subject to:} &
\end{array}$$

$$\begin{array}{rcll}
s_0 = 100, \quad b_0 = 0, & & x_0 = 0, & \text{[Initial sunday position]} \\
s_0 + b_1 & = & b_0 + 0.5x_0 + x_1 + s_1, & \text{[Mon]} \\
2x_0 + 1.01s_1 + b_2 & = & b_1 + 0.5x_1 + x_2 + s_2, & \text{[Tue]} \\
2x_1 + 1.01s_2 + b_3 & = & b_2 + 0.5x_2 + x_3 + s_3, & \text{[Wed]} \\
2x_2 + 1.01s_3 + b_4 & = & b_3 + 0.5x_3 + x_4 + s_4, & \text{[Thu]} \\
2x_3 + 1.01s_4 + b_5 & = & b_4 + 0.5x_4 + \ 0 + s_5, & \text{[Fri]}
\end{array}$$

$$x_j \ge 0 \text{ for } j = 1..4 \text{ and } s_j, b_j \ge 0 \text{ for } j = 1..5.$$

While comparing the alternative formulation to the original formulation in the previous question, we see a difference in terms of the time interest payments are made for the debt. In one formulation interest payments are paid simultaneously when the principal is paid, while in the other interest payments are deferred to be paid at the end of the horizon.
a) Express when the interest payments happen in both original and alternative formulations.
b) Do you expect to obtain a higher return with the original or with the alternative formulation? In other words, would you like to defer the interest payments? Explain. [*This question was inspired by Blair Flicker, OPRE 6302 student in Spring 2010.*]

16. Linear Regression with Linear Programs: Given data points $\{(x_i, y_i) : i = 1..N\}$, Linear Regression fits the line $y = ax + b$ to data points by minimizing the square of the distance between the data points and the line:

$$\text{Min} \quad \sum_{j=1}^{N} (y_j - ax_j - b)^2.$$

We define the rectilinear distance along the y axis and between two points $e = (e_x, e_y)$ and $f = (f_x, f_y)$ as $|e_y - f_y|$ where e_x, f_x and e_y, f_y are the x, y coordinates of the point e and f respectively. Note that this distance computes distances between two points only along the y axis (i.e. $|e_y - f_y|$) and disregards the distance along the x axis (i.e. $|e_x - f_x|$). For example, if $e = (3, 7)$ and $f = (4, 5)$, the rectilinear distance along y is $|e_y - f_y| = 2$.
a) Formulate an LP to find a line $y = ax + b$ closest (in rectilinear distance along y) to a given set of data points $\{(x_i, y_i) : i = 1..N\}$. Basically, replace the objective function of linear regression with

$$\text{Min} \quad \sum_{j=1}^{N} |y_j - ax_j - b|$$

and convert this into a series of expressions that can be included in an LP. You may assume that $N = 3$ if you are not comfortable with summations and indexing.
b) Find the optimal solution (the objective value and the equation of the line) by inspection when $N = 2$.

17. Manpower Planning: At the post office on the Coit street, each employee works exactly for 5 **consecutive** days per week. To provide a satisfactory customer service, the post office needs the following number of employees each day:

Days	Mon	Tue	Wed	Thu	Fri	Sat	Sun
# of employees required	9	6	5	8	11	13	4

a) Provide an LP formulation to minimize the number of employees.
b) Suppose some employees are willing to do overtime and work for one more day right after their 5 day regular schedule. Suppose that overtime labor rate is 50% more than regular labor rate. Provide an LP formulation to minimize the labor costs.

18. Market clearing model: Consider a market with I suppliers, J buyers and K products being bought and sold. A generic supplier is named as supplier i, a generic buyer is named buyer j and a generic product is named product k. Supplier i supplies at most S_{ik} units of product k and asks $\$A_{ik}$ price per unit of product k. Buyer j demands at most D_{jk} units of product k and bids $\$B_{jk}$ per unit of product k. Suppose that you work as an auction: You collect all the pricing and bidding information, then allocate supplies to buyers. This mechanism is called market clearing. You would be making profit from the spread between the bidding price B_{jk} and the asking price A_{ik} for every unit of product k sold from supplier i to buyer j.
a) Provide an LP formulation to maximize market clearing profit.
b) Suppose supplier i and buyer j are not at the same location and the available fleet capacity allows for transportation of at most U_{ij} products from supplier i to buyer j. In addition you incur a cost of $\$C_{ij}$ per unit of any product transported from supplier i to buyer j. Provide an LP formulation to maximize market clearing profit under these additional conditions.
c) Suppose that the fleet capacities U_{ij} of b) are given in terms of kilos and each product k weighs W_k kilos. Modify your formulation to b).

19. Road construction model: Suppose that we draw a line from the intersection of Coit & G. Bush to the intersection of Route 75 & G. Bush. Let h_i be the constant and known elevation (from the sea level) of the terrain on the line from the $i-1$st kilometer to the ith kilometer measured from the intersection of Coit & G. Bush, for $i = 1..N$. As a test of understanding the notation, note that N is the distance in kilometers between intersections and h_1 is the constant elevation of the line while traversing it from the intersection of Coit & G. Bush towards Route 75 for 1 kilometer. Further suppose that you want to construct a road between these two intersections by adding or removing terrain so as to make your road smooth. Actually, it is required that the slope of your road (in either direction) can not be larger than s. If you are willing to add and remove a lot of terrain, you can actually construct a level road all the way, needless to say this would require a huge budget. Provide an LP that will minimize the total terrain added or removed for the road construction and respect the slope constraint.

10 Solutions of Selected Problems

1. Let x = number of TV stands produced in a week. Let c = cost of producing x TV stands per week. I suggest that you draw the cost function c as x varies. You will see that c is composed of two lines

intersecting at $x = 10$. The equation of the lines are $c = 60x + 0$ and $c = 75x - 150$. The first line gives the cost for $x \leq 10$ and the second gives the cost for $x \geq 10$. Convince yourself that the cost can be written as $c = \max\{60x,\ 75x - 150\}$. In the formulation below we represent the maximum with inequalities, note that those inequalities are similar to the ones in the *Nathan's apartment cleaning example*.

$$
\begin{array}{lrrcrr}
\text{Max} & 100x & -c & & & \\
\text{Subject to:} & & & & & \\
 & 60x & -c & \leq & 0 & (1) \\
 & 75x & -c & \leq & 150 & (2) \\
 & & c & \leq & 1200 & (3) \\
 & x & & \geq & 2 & (4) \\
 & x, & c & \geq & 0 & .
\end{array}
$$

2. a) Suppose that Dr. Martin always keeps the same number of rabbits and chickens. Let r (c) = number of rabbits (chicken) raised at any time. The revenue obtained by raising a rabbit for a week is \$25/20 and the same number is \$15/18 for a chicken.

$$
\begin{array}{lrrcrr}
\text{Max} & (25/20)r & +(15/18)c & & & \\
\text{Subject to:} & & & & & \\
 & 2r & +0.5c & \leq & 12 & (1) \\
 & r & +0.6c & \leq & 5 & (2) \\
 & r, & c & \geq & 0 & .
\end{array}
$$

b) Add $r - 4c \geq 0$.

3. a) Let x_{ij} = dollar investment for route ij where i and j are the initials of Dallas, Plano, Addison or Richardson.

$$
\begin{array}{lrrrrcrr}
\text{Max} & x_{PD} & +1.2x_{RA} & +1.2x_{PR} & +1.1x_{RD} & & & \\
\text{Subject to:} & & & & & & & \\
 & x_{PD} & +x_{RA} & +x_{PR} & +x_{RD} & = & 1,000,000 & (1) \\
 & x_{PD} & & +x_{PR} & & \geq & 800,000 & (2) \\
 & x_{PD} & & & +x_{RD} & \geq & 300,000 & (3) \\
 & x_{PD}, & x_{RA}, & x_{PR}, & x_{RD}, & \geq & 0 & .
\end{array}
$$

b) Let $\bar{x}_{RA} > 0$ and $\bar{x}_{PR}$ be the value of optimal investment for Richardson - Addison route and Plano - Richardson route. Consider another solution where we modify only x_{RA} and x_{PR} as $x_{RA} = 0$ and $x_{PR} = \bar{x}_{PR} + \bar{x}_{RA}$. The new solution is feasible (why?) and has the same objective value. Thus, the new solution is optimal and it does not require investing for Richardson - Addison route.

11 LP Problems and Solutions from Plano High School

1. During slack time, the sawing and fabricating benches in Milam Cabinet Shop are used to make wooden hanging baskets and plant stands. The sawing bench has at most ten hours, and the fabricating bench has at most eight hours of slack time each week. Hanging baskets take one-fourth hour of sawing, while plant stands take one-third hour of sawing. Hanging baskets take one-half hour of fabricating, but plant stands take one-sixth hour of fabricating. Hanging baskets sell for a profit of \$6, and plant stands sell for a profit of \$8. Write a mathematical model to maximize profits.

H = # of Hanging baskets
P = # of Plant stands
Max 6H + 8P
s.t. $\frac{1}{4}H + \frac{1}{3}P \leq 10$
$\frac{1}{2}H + \frac{1}{6}P \leq 8$
$H \geq 0$
$P \geq 0$

2. The Holland family decides to raise and sell peppers and tomatoes to supplement their income. They have six acres of land. They believe they can make \$2,000 an acre on peppers and \$3,000 an acre on tomatoes. From past experience, they feel that they cannot take care of more than five acres of peppers or four acres of tomatoes. Write a mathematical model to show how many acres of each they should grow to maximize profit.

   P = # of acres of peppers
   T = # of acres of tomatoes
   Max 2000P + 3000T
   s.t. $P \leq 5$
   $T \leq 4$
   $P + T \leq 6$
   $P \geq 0$
   $T \geq 0$

3. Bonham Garden Fertilizers produce Regular and Super-Gro formulations. There are ten employees or 400 hours of production time each week. It takes one-fourth hour to produce and package either Regular or Super-Gro. Bonham has \$7,000 to spend on raw materials. Raw materials cost \$2 package for Regular, and \$5 per package for Super-Gro. Bonham makes \$1 profit on Regular and \$2 profit on Super-Gro per package. Write a mathematical model to show how many packages of each Bonham should formulate to maximize profit.

   R = # of packages of Regular
   S = # of packages of Super-Gro
   Max R + 2S
   s.t. $2R + 5S \leq 7000$
   $\frac{1}{4}(R + S) \leq 400$
   $R \geq 0$
   $S \geq 0$

4. Thomas & Brown Accounting audits books and prepares tax return. It employs three CPAs, each working 40 hours per week. The owners, after looking for new clients, have a total of 20 hours per week to review the work. The profit on an audit is \$400 and on a tax return \$125. An audit requires 6 hours of CPA time and 2 hours of review time. A tax return requires 3 hours of CPA time and one-fourth hour of review time. Write a mathematical model to show what mix of tasks they should do to maximize profits.

A = # of audits
T = # of tax returns
Max 400A + 125T
s.t. $6A + 3T \leq 120$
$2A + \frac{1}{4}T \leq 20$
$A \geq 0$
$T \geq 0$

5. Jim Fannin Pharmaceuticals sells to drug stores. As an independent jobber, he can choose his own territory. A call on a small-town drug store usually results in a $1,200 sale, but takes an average of six hours. A call on a big-city drug store usually results in a $750 sale but only takes an average of two hours. Jim doesnt work more than 42 hours a week. Write a mathematical model to show how he should spend his time calling on customers to maximize sales.

   S = # of calls in small town
   B = # of calls in big city
   Max 1200S + 750B
   s.t. $6S + 2B \leq 42$
   $S \geq 0$
   $B \geq 0$

6. The Biltrite Furniture Company makes wooden desks and chairs. Carpenters and finishers work on each item. On the average the carpenter spends 4 hours working on each chair and 8 hours working on each desk. There are enough carpenters for at most 8000 worker-hours per week. The finishers spend about 2 hours on each chair and 1 hour on each desk. There are enough finishers for up to 1300 worker-hours per week. If there is a profit of $80 per chair and $135 per desk what production level will maximize the profit?

   C = # of chairs produced
   D = # of desks produced
   Max 80C + 135D
   s.t. $4C + 8D \leq 8000$
   $2C + D \leq 1300$
   $C \geq 0$
   $D \geq 0$

7. Jocelyns Jewelry Store makes rings and pendants. Every week the staff uses at most 500 gm of metal and spends at most 150 hours making jewelry. It takes 5 gm of metal to make a ring and 20 gm to make a pendant. Each ring takes 2 hours and each pendant requires 3 hours to make. The profit on each ring is $70 and the profit on each pendant is $90. How many of each should the store make to maximize its profit?

   R = # of rings made
   P = # of pendants made
   Max 70R + 90P

s.t. $5R + 20P \leq 500$
$2R + 3P \leq 150$
$R \geq 0$
$P \geq 0$

8. Major Motors must produce at least 5,000 luxury cars and 12,000 medium-prices cars. They must also produce at most 30,000 compact cars. The company owns two factories A and B at different locations. Factory A produces 20, 40 and 60 units of luxury, medium and compact cars, respectively each day. Factory B produces 10, 30 and 50, respectively, each day. If factory A costs $960,000 per day to operate and B costs $750,000 per day, find the number of days each should operate to minimize the costs yet meet the requirements for car production. What is the minimum cost?

   A = # of days factory A operates
   B = # of days factory B operates
   Min 960000A + 750000B
   s.t. $20A + 10B \geq 5000$
   $40A + 30B \geq 12000$
   $60A + 50B \leq 30000$
   $A \geq 0$
   $B \geq 0$

9. Dr. Delphinium Gardening Supplies contracts, on a weekly basis, for suppliers for its stores. Clay Corner can provide 150 glazed and 400 unglazed flowerpots per week. It can commit to at most 15 weeks of production. The contract is for $500 per week. Wheel Works can provide 50 glazed and 100 unglazed flowerpots per week. It can commit to at most 35 weeks of production. This contract is for $250 per week. To satisfy existing orders for spring shipment of plants, Dr. Delphinium needs at least 2250 glazed and 5000 unglazed flowerpots. How many weeks of production from each supplier should be contracted to minimize costs? What is the minimum cost?

   C = # of weeks of production from Clay Corner
   W = # of weeks of production from Wheel Works
   Min 500C + 250W
   s.t. $150C + 50W \geq 2250$
   $400C + 100W \geq 5000$
   $C \leq 15$
   $W \leq 35$
   $C \geq 0$
   $W \geq 0$

10. Ruby Sappire Culpepper, who is into being fit, takes vitamin pills. Each day she must have at least 16 units of vitamin A, and at least 5 units of vitamin B_1, and at most 20 units of vitamin C. She can choose between pill 1 which contains 8 units of A, 1 unit of B_1, and 2 units of C. Pill 2 contains 2 units of A, 1 unit of B_1, and 2 units of C. Pill 1 costs 15 cents and pill 2 costs 30 cents. How many of each pill should she buy in order to minimize her cost? What is that cost?

X = # of pill 1 she buys
Y = # of pill 2 she buys
Min 15X + 30Y
s.t. $8X + 2Y \geq 16$
$X + Y \geq 5$
$2X + 2Y \leq 20$
$X \geq 0$
$Y \geq 0$