

# Statistical Methods in AI and ML

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A powerful and flexible set of tools for modeling  
problems in AI/ML

Judea Pearl won the Turing award for his work on  
Bayesian networks!  
(among other achievements)

Exploit **locality** and structural features of a given model in order to gain insight about **global properties**

- What this course is:
  - Probabilistic graphical models
  - Topics:
    - representing data
    - exact and approximate statistical inference
    - model learning
    - variational methods in ML

- What you should be able to do at the end:
  - Design statistical models for applications in your domain of interest
  - Apply learning and inference algorithms to solve real problems (exactly or approximately)
  - Understand the complexity issues involved in the modeling decisions and algorithmic choices

# Prerequisites



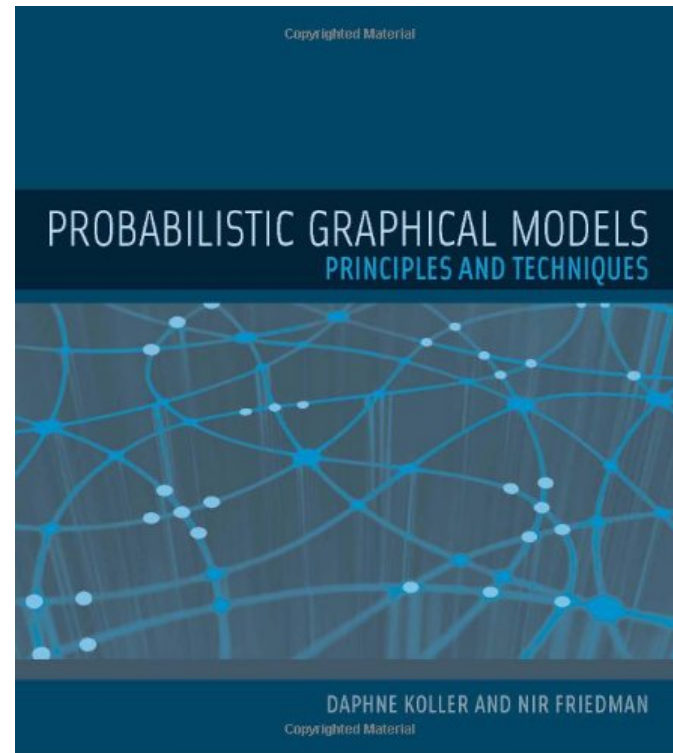
- CS 5343: Algorithm Analysis and Data Structures
- CS 3341: Probability and Statistics in Computer Science and Software Engineering
- Basically, comfort with probability and algorithms (machine learning is helpful, but not required)

# Suggested Textbook



Readings will be posted  
online before each  
lecture

Check the course website  
for additional resources  
and papers



- In addition, some lecture notes, in book format, will be made available for the main topics
- The idea is to build a set of notes that aligns well with the presentation of course material
- Comments, suggestions, corrections are welcome/encouraged



- 4-6 problem sets (70%)
  - See collaboration policy on the web
- Final project (25%)
- Class/Piazza participation & extra credit (5%)

*-subject to change-*

- Instructor: Nicholas Ruozzi
  - Office: ECSS 3.409
  - Office hours: M. 1pm – 2pm, W. 10am-11am, and by appointment
- TA: Shahab Shams
  - Office hours and location TBD
- Course website:  
<http://www.utdallas.edu/~nrr150130/cs6347/2019sp/>

- Model the world (or at least the problem) as a collection of random variables related through some joint probability distribution
  - Compactly represent the distribution
  - Undirected graphical models
  - Directed graphical models
- Learn the distribution from observed data
  - Maximum likelihood, SVMs, etc.
- Make predictions (statistical inference)

# Inference and Learning

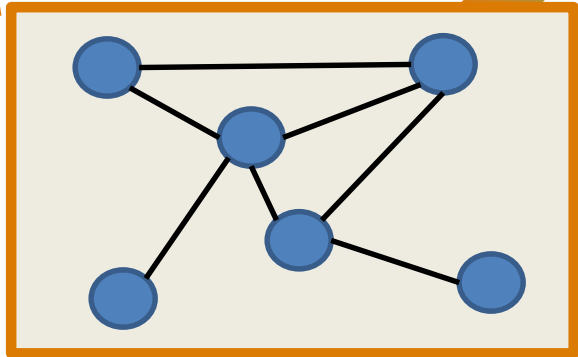


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**Collect Data**

$$Z(\theta) = \sum_x p(x; \theta)$$

**Use the model to do inference / make predictions**



**“Learn” a model that represents the observed data**

# Inference and Learning

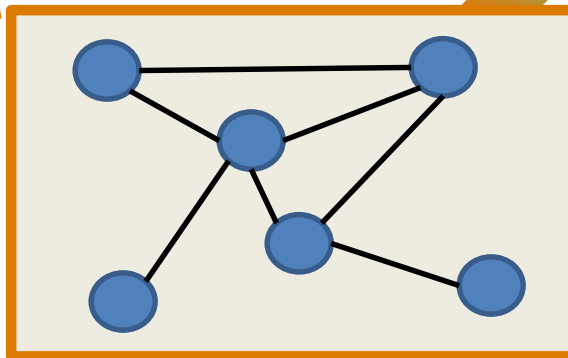


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**Data sets can be large**

$$Z(\theta) = \sum_x p(x; \theta)$$

**Inference needs to be fast**



**Data must be compactly modeled**

# Applications



- Computer vision
- Natural language processing
- Robotics
- Computational biology
- Computational neuroscience
- Text translation
- Text-to-speech
- Many more...

- A graphical model is a graph together with "local interactions"
- The graph and interactions model a global optimization or learning problem
- The study of graphical models is concerned with how to exploit local structure to solve these problems either exactly or approximately

# Probability Review



- **Sample space** specifies the set of possible outcomes
  - For example,  $\Omega = \{H, T\}$  would be the set of possible outcomes of a coin flip
- Each element  $\omega \in \Omega$  is associated with a number  $p(\omega) \in [0,1]$  called a **probability**

$$\sum_{\omega \in \Omega} p(\omega) = 1$$

- For example, a biased coin might have  $p(H) = .6$  and  $p(T) = .4$

- An **event** is a subset of the sample space
  - Let  $\Omega = \{1, 2, 3, 4, 5, 6\}$  be the 6 possible outcomes of a dice role
  - $A = \{1, 5, 6\} \subseteq \Omega$  would be the event that the dice roll comes up as a one, five, or six
- The probability of an event is just the sum of all of the outcomes that it contains
  - $p(A) = p(1) + p(5) + p(6)$

# Independence

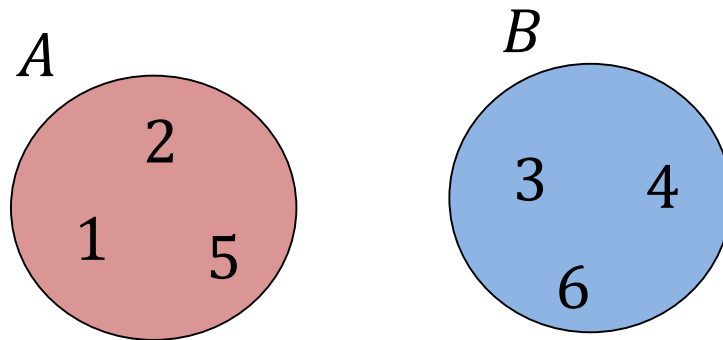


- Two events  $A$  and  $B$  are **independent** if

$$p(A \cap B) = p(A)p(B)$$

Let's suppose that we have a fair die:  $p(1) = \dots = p(6) = 1/6$

If  $A = \{1, 2, 5\}$  and  $B = \{3, 4, 6\}$  are  $A$  and  $B$  independent?



# Independence

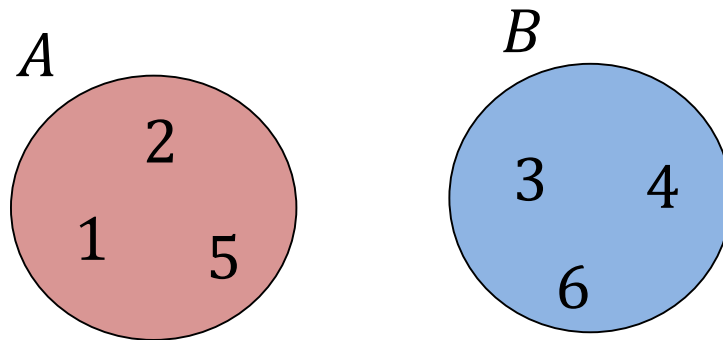


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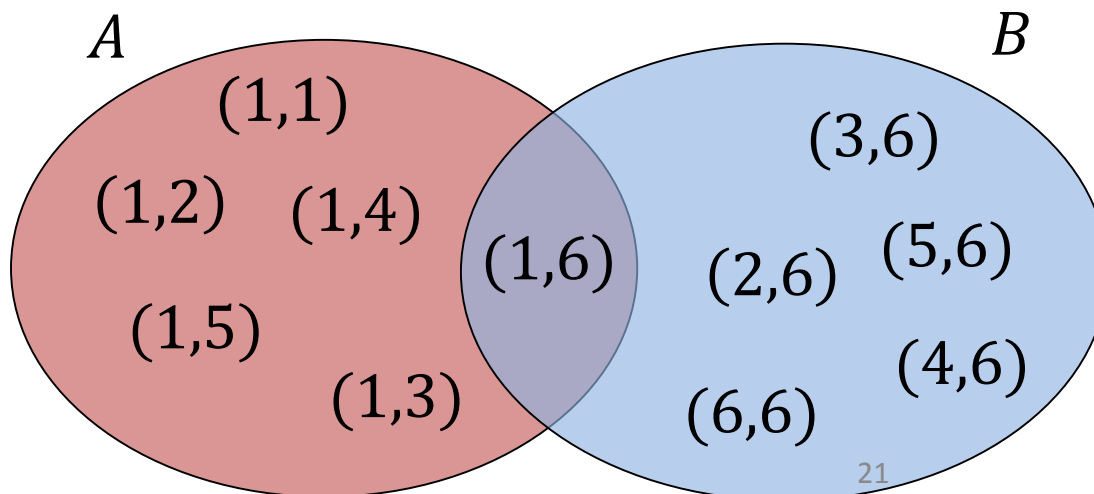
*No!*

$$p(A \cap B) = 0 \neq \frac{1}{4}$$

# Independence



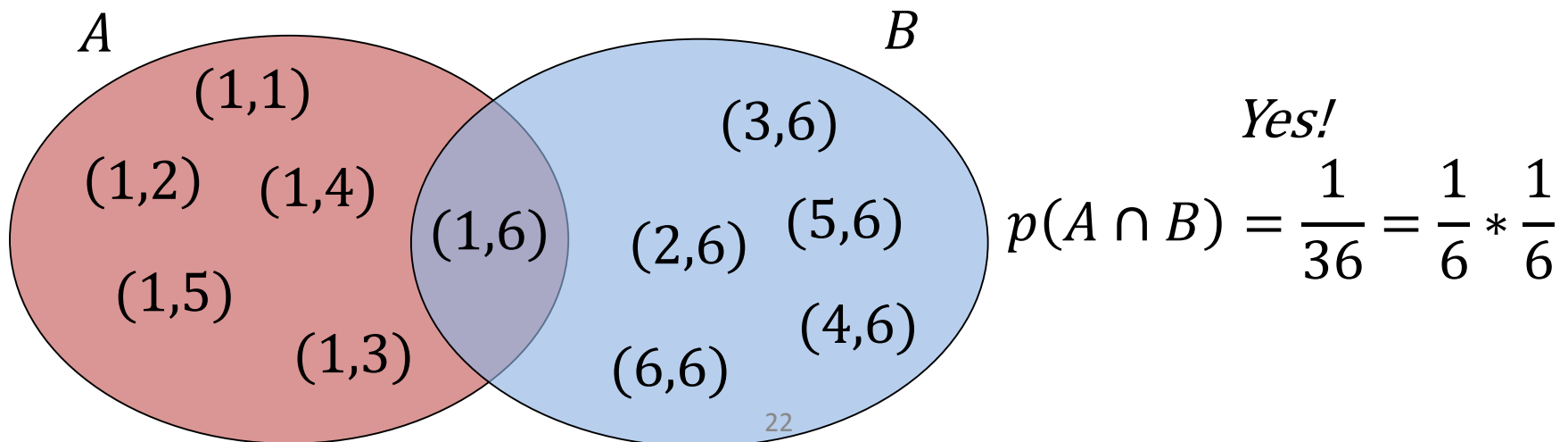
- Now, suppose that  $\Omega = \{(1,1), (1,2), \dots, (6,6)\}$  is the set of all possible rolls of two **unbiased** dice
- Let  $A = \{(1,1), (1,2), (1,3), \dots, (1,6)\}$  be the event that the first die is a one and let  $B = \{(1,6), (2,6), \dots, (6,6)\}$  be the event that the second die is a six
- Are  $A$  and  $B$  independent?



# Independence



- Now, suppose that  $\Omega = \{(1,1), (1,2), \dots, (6,6)\}$  is the set of all possible rolls of two **unbiased** dice
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- Are  $A$  and  $B$  independent?



- The **conditional probability** of an event  $A$  given an event  $B$  with  $p(B) > 0$  is defined to be

$$p(A|B) = \frac{p(A \cap B)}{P(B)}$$

- This is the probability of the event  $A \cap B$  over the sample space  $\Omega' = B$
- Some properties:
  - $\sum_{\omega \in B} p(\omega|B) = 1$
  - If  $A$  and  $B$  are independent, then  $p(A|B) = p(A)$

# Simple Example



Cheated	Grade	Probability
Yes	A	.15
Yes	F	.05
No	A	.5
No	F	.3



# Simple Example



Cheated	Grade	Probability
Yes	A	.15
Yes	F	.05
No	A	.5
No	F	.3

$$p(\textit{Cheated} = \textit{Yes} \mid \textit{Grade} = \textit{F}) = ?$$

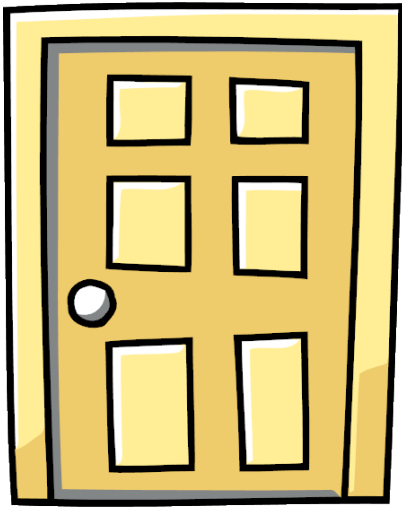
# Simple Example



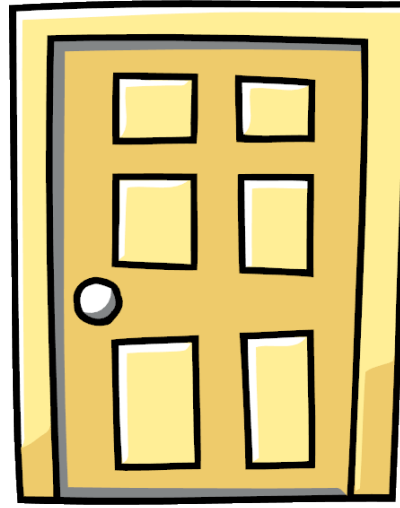
Cheated	Grade	Probability
Yes	A	.15
Yes	F	.05
No	A	.5
No	F	.3

$$p(\text{Cheated} = \text{Yes} | \text{Grade} = \text{F}) = \frac{.05}{.35} \approx .14$$

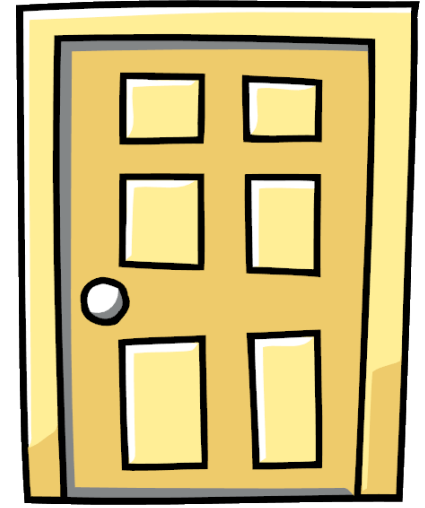
# The Monty Hall Problem



**1**



**2**



**3**

$$p(A \cap B) = p(A)p(B|A)$$

$$\begin{aligned} p(A \cap B \cap C) &= p(A \cap B)p(C|A \cap B) \\ &= p(A)p(B|A)p(C|A \cap B) \end{aligned}$$

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$$p\left(\bigcap_{i=1}^n A_i\right) = p(A_1)p(A_2|A_1) \dots p(A_n|A_1 \cap \dots \cap A_{n-1})$$

# Conditional Independence



- Two events  $A$  and  $B$  are independent if learning something about  $B$  tells you nothing about  $A$  (and vice versa)
- Two events  $A$  and  $B$  are **conditionally independent** given  $C$  if

$$p(A \cap B|C) = p(A|C)p(B|C)$$

- This is equivalent to

$$p(A|B \cap C) = p(A|C)$$

- That is, given  $C$ , information about  $B$  tells you nothing about  $A$  (and vice versa)

# Conditional Independence



- Let  $\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$  be the outcomes resulting from tossing two different fair coins
- Let  $A$  be the event that the first coin is heads
- Let  $B$  be the event that the second coin is heads
- Let  $C$  be the event that both coins are heads or both are tails
- $A$  and  $B$  are independent, but  $A$  and  $B$  are not independent given  $C$

- A discrete **random variable**,  $X$ , is a function from the state space  $\Omega$  into a discrete space  $D$

- For each  $x \in D$ ,

$$p(X = x) \equiv p(\{\omega \in \Omega : X(\omega) = x\})$$

is the probability that  $X$  takes the **value**  $x$

- $p(X)$  defines a probability distribution

- $\sum_{x \in D} p(X = x) = 1$

- Random variables partition the state space into disjoint events

# Example: Pair of Dice



- Let  $\Omega$  be the set of all possible outcomes of rolling a pair of dice
- Let  $p$  be the uniform probability distribution over all possible outcomes in  $\Omega$
- Let  $X(\omega)$  be equal to the sum of the value showing on the pair of dice in the outcome  $\omega$ 
  - $p(X = 2) = ?$
  - $p(X = 8) = ?$



# Example: Pair of Dice



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- Let  $p$  be the uniform probability distribution over all possible outcomes in  $\Omega$
- Let  $X(\omega)$  be equal to the sum of the value showing on the pair of dice in the outcome  $\omega$ 
  - $p(X = 2) = \frac{1}{36}$
  - $p(X = 8) = ?$

# Example: Pair of Dice



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- Let  $p$  be the uniform probability distribution over all possible outcomes in  $\Omega$
- Let  $X(\omega)$  be equal to the sum of the value showing on the pair of dice in the outcome  $\omega$ 
  - $p(X = 2) = \frac{1}{36}$
  - $p(X = 8) = \frac{5}{36}$

- We can have vectors of random variables as well

$$X(\omega) = [X_1(\omega), \dots, X_n(\omega)]$$

- The **joint distribution** is  $p(X_1 = x_1, \dots, X_n = x_n)$  is

$$p(X_1 = x_1 \cap \dots \cap X_n = x_n)$$

typically written as

$$p(x_1, \dots, x_n)$$

- Because  $X_i = x_i$  is an event, all of the same rules - independence, conditioning, chain rule, etc. - still apply

- Two random variables  $X_1$  and  $X_2$  are independent if

$$p(X_1 = x_1, X_2 = x_2) = p(X_1 = x_1)p(X_2 = x_2)$$

for all values of  $x_1$  and  $x_2$

- Similar definition for conditional independence
- The conditional distribution of  $X_1$  given  $X_2 = x_2$  is

$$p(X_1 | X_2 = x_2) = \frac{p(X_1, X_2 = x_2)}{p(X_2 = x_2)}$$

this means that this relationship holds for all choices of  $x_1$

# Expected Value



- The **expected value** of a real-valued random variable is the weighted sum of its outcomes

$$E[X] = \sum_{x \in D} p(X = d) \cdot d$$

- Expected value is linear

$$E[X + Y] = E[X] + E[Y]$$

# Expected Value: Lotteries



- Powerball Lottery currently has a grand prize of \$112 million
- Odds of winning the grand prize are  $1/292,201,338$
- Tickets cost \$2 each
- Expected value of the game

$$= \frac{-2 \cdot 292,201,337}{292,201,338} + \frac{112,000,000 - 2}{292,201,338} \approx \$.38$$

- The **variance** of a random variable measures its squared deviation from its mean

$$\text{var}(X) = E[(X - E[X])^2]$$

- Estimates the square of the expected amount by which a random variable deviates from its expected value