Logistic Regression

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based on the slides of Vibhav Gogate

Last Time

- Supervised learning via naive Bayes
  - Use MLE to estimate a distribution
  - Classify by looking at the conditional distribution, $p(y|x)$
- Today: logistic regression

Naïve Bayes

- Naïve Bayes assumes that the underlying distribution that generated the data satisfies the following conditional independence assumption

$$p(x_1, \dots, x_m | y) = \prod_{i=1}^{m} p(x_i | y)$$

- Use MLE to learn the "best" model that satisfies this assumption

- Classify via

$$y^* = \arg\max_y p(y) p(x_1, \dots, x_m | y)$$
$$= \arg\max_y p(y) \prod_i^m p(x_i | y)$$

MLE for the Parameters of NB

- Given dataset, count occurrences for all pairs
  - $Count(X_i = x_i, Y = y)$ is the number of samples in which $X_i = x_i$ and $Y = y$
- MLE for discrete NB

$$p(Y = y) = \frac{Count(Y = y)}{\sum_{y'} Count(Y = y')}$$

$$p(X_i = x_i | Y = y) = \frac{Count(X_i = x_i, Y = y)}{\sum_{x_i'} Count\,(X_i = x_i', Y = y)}$$

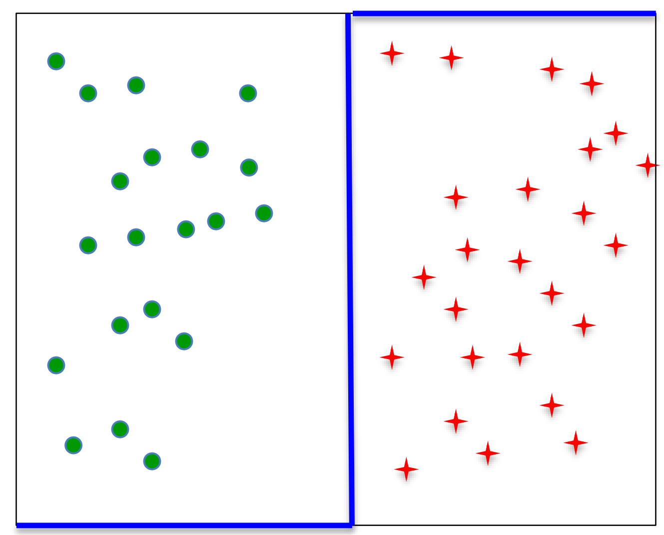
Naïve Bayes Calculations

Day	Outlook	Temperature	Humidity	Wind	PlayTennis
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

Logistic Regression

- Learn $p(Y|X)$ directly from the data
  - Assume a particular functional form, e.g., a linear classifier $p(Y = 1|x) = 1$ on one side and 0 on the other
  - Not differentiable…
    - Makes it difficult to learn
    - Can't handle noisy labels

$p(Y = 1|x) = 0$



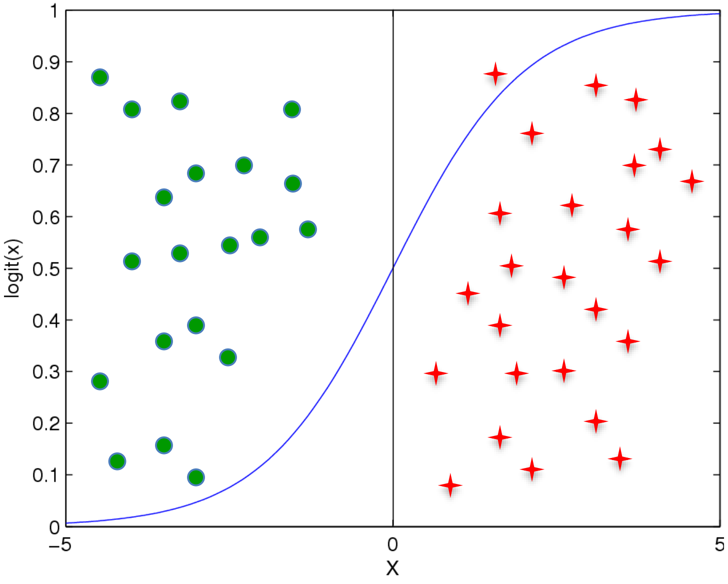
$p(Y = 1|x) = 1$

Logistic Regression

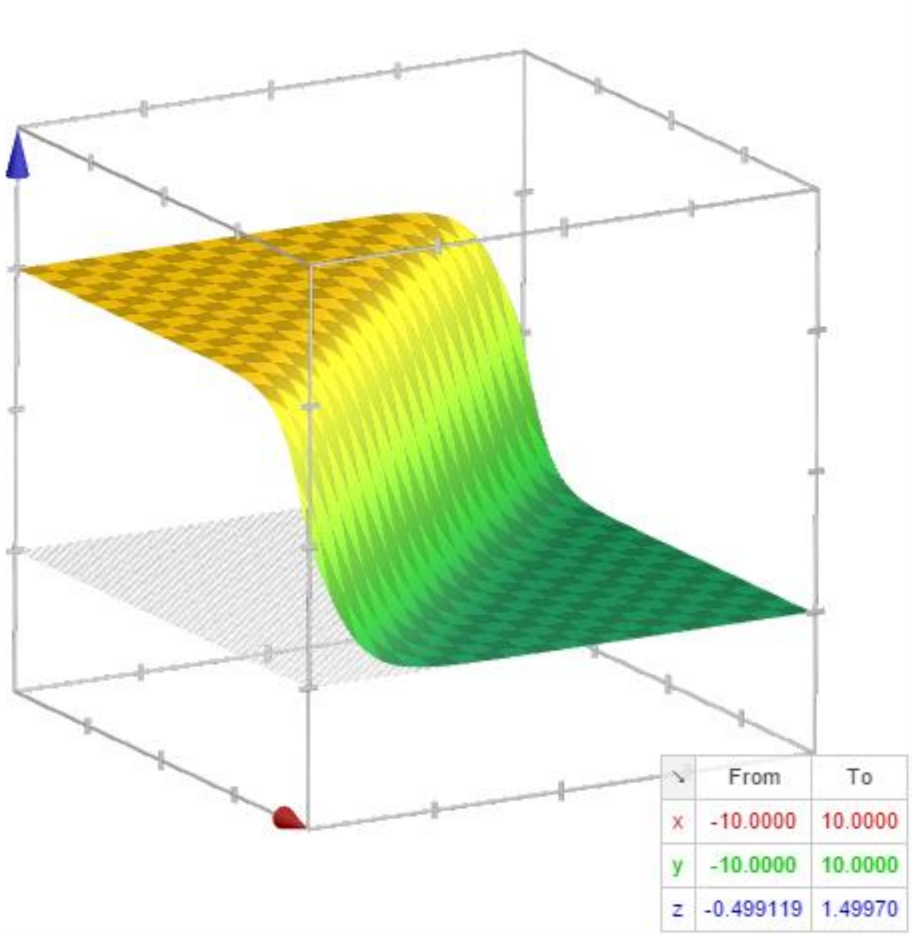
- Learn $p(y|x)$ directly from the data
  - Assume a particular functional form

$$p(Y = -1|x) = \frac{1}{1 + \exp(w^T x + b)}$$

$$p(Y = 1|x) = \frac{\exp(w^T x + b)}{1 + \exp(w^T x + b)}$$



Logistic Function in m Dimensions



$$p(Y = -1|x) = \frac{1}{1 + \exp(w^T x + b)}$$

Can be applied to discrete and continuous features

Functional Form: Two classes

- Given some w and b, we can classify a new point x by assigning the label 1 if $p(Y = 1|x) > p(Y = -1|x)$ and -1 otherwise
  - This leads to a linear classification rule:
    - Classify as a 1 if $w^T x + b > 0$
    - Classify as a -1 if $w^T x + b < 0$

Learning the Weights

- To learn the weights, we maximize the conditional likelihood

$$(w^*, b^*) = \arg\max_{w,b} \prod_{i=1}^{N} p(y^{(i)} | x^{(i)}, w, b)$$

- This is the not the same strategy that we used in the case of naive Bayes
  - For naive Bayes, we maximized the log-likelihood, not the conditional log-likelihood

Generative vs. Discriminative Classifiers

Generative classifier:
(e.g., Naïve Bayes)

- Assume some **functional form** for $p(x|y), p(y)$
- Estimate parameters of $p(x|y)$, $p(y)$ directly from training data
- Use Bayes rule to calculate $p(y|x)$
- This is a **generative model**
  - **Indirect** computation of $p(Y|X)$ through Bayes rule
  - As a result, **can also generate a sample of the data,** $p(x) = \sum_y p(y)p(x|y)$

Discriminative classifiers:
(e.g., Logistic Regression)

- Assume some **functional form for** $p(y|x)$
- Estimate parameters of $p(y|x)$ directly from training data
- This is the **discriminative model**
  - Directly learn $p(y|x)$
  - But **cannot obtain a sample of the data** as $p(x)$ is not available
  - Useful for discriminating labels

Learning the Weights

$$
\begin{aligned}
\ell(w,b) &= \ln \prod_{i=1}^{N} p(y^{(i)}|x^{(i)},w,b) \\
&= \sum_{i=1}^{N} \ln p(y^{(i)}|x^{(i)},w,b) \\
&= \sum_{i=1}^{N} \frac{y^{(i)}+1}{2} \ln p(Y=1|x^{(i)},w,b) + \left(1 - \frac{y^{(i)}+1}{2}\right) \ln p(Y=-1|x^{(i)},w,b) \\
&= \sum_{i=1}^{N} \frac{y^{(i)}+1}{2} \ln \frac{p\left(Y=1\middle|x^{(i)},w,b\right)}{p\left(Y=-1\middle|x^{(i)},w,b\right)} + \ln p(Y=-1|x^{(i)},w,b) \\
&= \sum_{i=1}^{N} \frac{y^{(i)}+1}{2} \left(w^T x^{(i)} + b\right) - \ln\left(1 + \exp\left(w^T x^{(i)} + b\right)\right)
\end{aligned}
$$

Learning the Weights

$$
\begin{aligned}
\ell(w,b) &= \ln \prod_{i=1}^{N} p(y^{(i)}|x^{(i)},w,b) \\
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&= \sum_{i=1}^{N} \frac{y^{(i)}+1}{2} (w^T x^{(i)} + b) - \ln\left(1+\exp\left(w^T x^{(i)} + b\right)\right)
\end{aligned}
$$

This is concave in over w and b: take derivatives and solve!

Learning the Weights

$$\begin{aligned}
\ell(w,b) \quad &= \ln \prod_{i=1}^{N} p(y^{(i)}|x^{(i)},w,b) \\
&= \sum_{i=1}^{N} \ln p(y^{(i)}|x^{(i)},w,b) \\
&= \sum_{i=1}^{N} \frac{y^{(i)}+1}{2} \ln p(Y=1|x^{(i)},w,b) + \left(1 - \frac{y^{(i)}+1}{2}\right) \ln p(Y=-1|x^{(i)},w,b) \\
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&= \sum_{i=1}^{N} \frac{y^{(i)}+1}{2} \left(w^T x^{(i)} + b\right) - \ln\left(1+\exp\left(w^T x^{(i)} + b\right)\right)
\end{aligned}$$

No closed form solution ☹

Learning the Weights

- Can apply gradient ascent to maximize the conditional likelihood

$$\frac{\partial \ell}{\partial b} = \sum_{i=1}^{N} \left[\frac{y^{(i)} + 1}{2} - p(Y = 1 | x^{(i)}, w, b)\right]$$

$$\frac{\partial \ell}{\partial w_j} = \sum_{i=1}^{N} x_j^{(i)} \left[\frac{y^{(i)} + 1}{2} - p(Y = 1 | x^{(i)}, w, b)\right]$$

Priors

- Can define priors on the weights and bias to prevent overfitting
  - Normal distribution, zero mean, identity covariance

$$p(w) = \prod_j \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{w_j^2}{2\sigma^2}\right)$$

  - "Pushes" parameters towards zero
- Regularization
  - Helps avoid very large weights and overfitting

Priors as Regularization

- The log-MAP objective with this Gaussian prior is then

$$\ln \prod_{i=1}^{N} p\left(y^{(i)} \middle| x^{(i)}, w, b\right) p(w) p(b) = \left[\sum_{i}^{N} \ln p\left(y^{(i)} \middle| x^{(i)}, w, b\right)\right] - \frac{\lambda}{2}\left(\|w\|_2^2 + b^2\right)$$

  – Quadratic penalty: drives weights towards zero

  – Adds a negative linear term to the gradients

  – Different priors can produce different kinds of regularization

Priors as Regularization

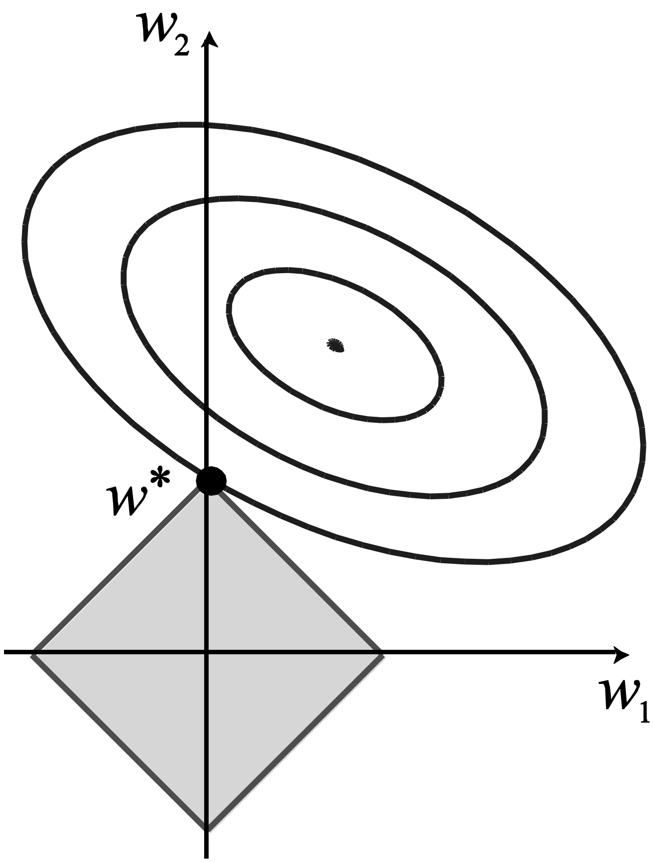
- The log-MAP objective with this Gaussian prior is then

$$\ln \prod_{i=1}^{N} p\left(y^{(i)} \middle| x^{(i)}, w, b\right) p(w)p(b) = \left[\sum_{i}^{N} \ln p\left(y^{(i)} \middle| x^{(i)}, w, b\right)\right] - \frac{\lambda}{2}\left(\|w\|_2^2 + b^2\right)$$

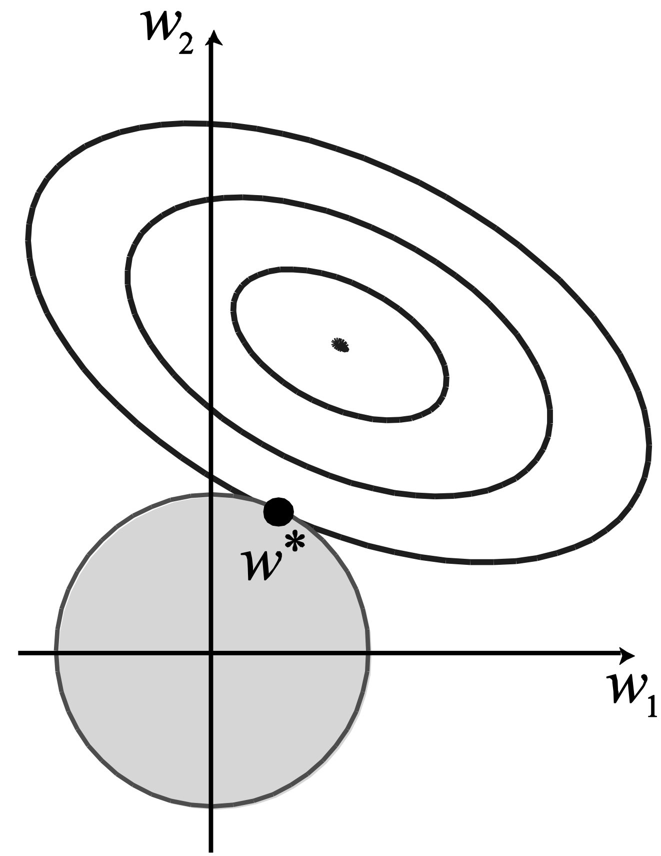
Somtimes called an ℓ_2 regularizer

  - Quadratic penalty: drives weights towards zero
  - Adds a negative linear term to the gradients
  - Different priors can produce different kinds of regularization

Regularization



ℓ_1



ℓ_2

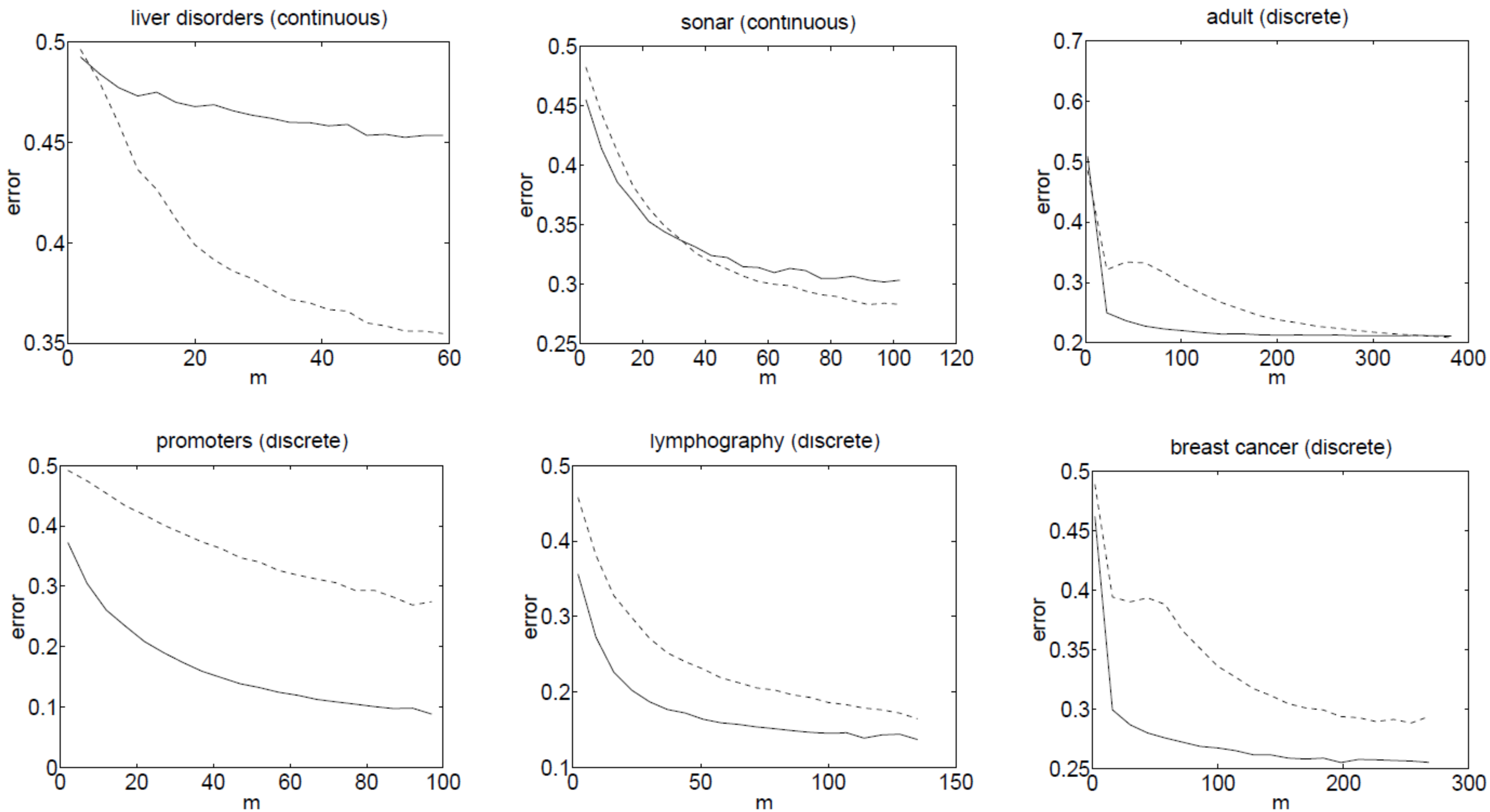
Naïve Bayes vs. Logistic Regression

- Generative vs. Discriminative classifiers
- Non-asymptotic analysis (for Gaussian NB)
  - Convergence rate of parameter estimates, (m = # of attributes in X)
    - Size of training data to get close to infinite data solution
    - Naïve Bayes needs $O(\log m)$ samples
      - NB converges quickly to its (perhaps less helpful) asymptotic estimates
    - Logistic Regression needs $O(m)$ samples
      - LR converges slower but makes no independence assumptions (typically less biased)

[Ng & Jordan, 2002]

UTD

NB vs. LR (on UCI datasets)



Naïve bayes

Logistic Regression

Sample size m

[Ng & Jordan, 2002]

LR in General

- Suppose that $y \in \{1, \ldots, R\}$, i.e., that there are R different class labels

- Can define a collection of weights and biases as follows

  – Choose a vector of biases and a matrix of weights such that for $y \neq R$

$$p(Y = k|x) = \frac{\exp(b_k + \sum_i w_{ki} x_i)}{1 + \sum_{j<R} \exp\left(b_j + \sum_i w_{ji} x_i\right)}$$

and

$$p(Y = R|x) = \frac{1}{1 + \sum_{j<R} \exp\left(b_j + \sum_i w_{ji} x_i\right)}$$