MORE
Learning Theory

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Based on the slides of Vibhav Gogate and David Sontag

Last Time

- Probably approximately correct (PAC)
  - The only reasonable expectation of a learner is that with high probability it learns a close approximation to the target concept
  - Specify two small parameters, $0 < \epsilon, \delta < 1$
    - ϵ is the error of the approximation
    - $(1 - \delta)$ is the probability that, given m i.i.d. samples, our learning algorithm produces a classifier with error at most ϵ

Learning Theory

- We use the observed data in order to learn a classifier
- Want to know how far the learned classifier deviates from the (unknown) underlying distribution
  - With too few samples, we will with high probability learn a classifier whose true error is quite high even though it may be a perfect classifier for the observed data
  - As we see more samples, we pick a classifier from the hypothesis space with low training error & hope that it also has low true error
    - Want this to be true with high probability – can we bound how many samples that we need?

Haussler, 1988

- What we proved last time:

Theorem: For a finite hypothesis space, H, with m i.i.d. samples, and $0 < \epsilon < 1$, the probability that any consistent classifier has true error larger than ϵ is at most $|H|e^{-\epsilon m}$

- We can turn this into a sample complexity bound

Sample Complexity

- Let δ be an upper bound on the desired probability of not ϵ-exhausting the sample space
  - The probability that the version space is not ϵ-exhausted is at most $|H|e^{-\epsilon m} \leq \delta$
  - Solving for m yields

$$m \geq -\frac{1}{\epsilon}\log\frac{\delta}{|H|}$$
$$= \left(\log|H| + \log\frac{1}{\delta}\right)/\epsilon$$

Generalizations

- How do we handle the case that there is no consistent classifier?
  - Pick the hypothesis with the lowest error on the training set, bound?
- What do we do if the hypothesis space isn't finite?
  - Infinite sample complexity?
  - Need a way to measure the complexity of the space that isn't based on its size

Chernoff Bounds

- **Chernoff bound**: Suppose $Y_1, \ldots, Y_m$ are i.i.d. random variables taking values in $\{0, 1\}$ such that $E_p[Y_i] = y$. For $\epsilon > 0$,

$$p\left(y - \frac{1}{m}\sum_i Y_i \geq \epsilon\right) \leq e^{-2m\epsilon^2}$$

Chernoff Bounds

- For $h \in H$, let Z_i^h be an indicator random variable that is one if h misclassifies the i^{th} data point

$$p(Z_i^h = 1) = \sum_{x,y} p(x,y) 1_{h(x) \neq y} = \epsilon_h$$

- Applying Chernoff bound to $Z_1^h, \dots, Z_m^h$ gives

$$p\left(\epsilon_h - \frac{1}{m}\sum_i Z_i^h \geq \epsilon\right) \leq e^{-2m\epsilon^2}$$

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$$p\left(\epsilon_h - \frac{1}{m}\sum_i Z_i^h \geq \epsilon\right) \leq e^{-2m\epsilon^2}$$

This is the training error

PAC Bounds

Theorem: For a finite hypothesis space H, m i.i.d. samples, and $0 < \epsilon < 1$, the probability that true error of any of the best classifiers (i.e., lowest training error) is larger than its training error plus ϵ is at most $|H|e^{-2m\epsilon^2}$

- Sample complexity (for desired $\delta \geq |H|e^{-2m\epsilon^2}$)

$$m \geq \left(\log|H| + \log\frac{1}{\delta}\right)/2\epsilon^2$$

PAC Bounds

- If we require that the previous error is bounded above by δ, then with probability $(1-\delta)$, for all $h \in H$

$$\epsilon_h \leq \underbrace{\epsilon_h^{train}}_{\text{"bias"}} + \underbrace{\sqrt{\frac{1}{2m}\left(\log|H| + \log\frac{1}{\delta}\right)}}_{\text{"variance"}}$$

  – For small $|H|$
    - High bias (may not be enough hypotheses to choose from)
    - Low variance

PAC Bounds

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$$\epsilon_h \leq \underbrace{\epsilon_h^{train}}_{\text{“bias”}} + \underbrace{\sqrt{\frac{1}{2m}\left(\log|H| + \log\frac{1}{\delta}\right)}}_{\text{“variance”}}$$

  – For large $|H|$
    - Low bias (lots of good hypotheses)
    - High variance

VC Dimension

- Our analysis for the finite case was based on $|H|$
  - This translates into infinite sample complexity
  - We can derive a different notion of complexity for infinite hypothesis spaces by considering only the number of points that can be correctly classified by some member of H

VC Dimension

- How many points in 1-D can be correctly classified by a linear separator?
  - 2 points:

Yes!

VC Dimension

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VC Dimension

- How many points in 1-D can be correctly classified by a linear separator?
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Yes!

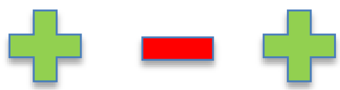
VC Dimension

- How many points in 1-D can be correctly classified by a linear separator?
  - 3 points:

Yes!

VC Dimension

- How many points in 1-D can be correctly classified by a linear separator?
  - 3 points:



NO!

VC Dimension

- How many points in 1-D can be correctly classified by a linear separator?
  - 3 points:

NO!

  - 3 points and up: for any collection of three or more there is always some choice of pluses and minuses such that that the points cannot be classified with a linear separator

VC Dimension

- A set of points is shattered by a hypothesis space H if and only if for every partition of the set of points into positive and negative examples, there exists some consistent $h \in H$

- The Vapnik–Chervonenkis (VC) dimension of H over inputs from X is the size of the ***largest*** finite subset of X shattered by H

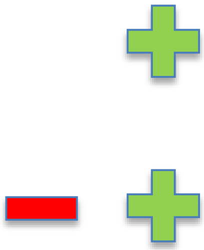
VC Dimension

- Common misconception:
  - VC dimension is determined by the largest shattered set of points, not the highest number such that all sets of points that size can be shattered

Cannot be shattered by a line

VC Dimension

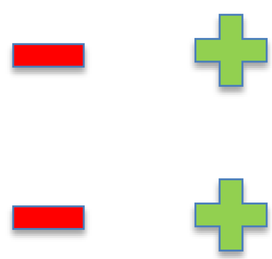
- Common misconception:
  - VC dimension is determined by the largest shattered set of points, not the highest number such that all sets of points that size can be shattered



Can be shattered by a line (no matter the labels), so VC dimension is at least 3

VC Dimension

- What is the VC dimension of 2-D space under linear separators?
  - It is at least three from the last slide
  - Can some set of four points be shattered?



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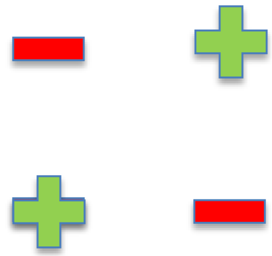


VC Dimension

- What is the VC dimension of 2-D space under linear separators?
  - It is at least three from the last slide
  - Can some set of four points be shattered?

NO! This means that the VC dimension is at most 3

VC Dimension

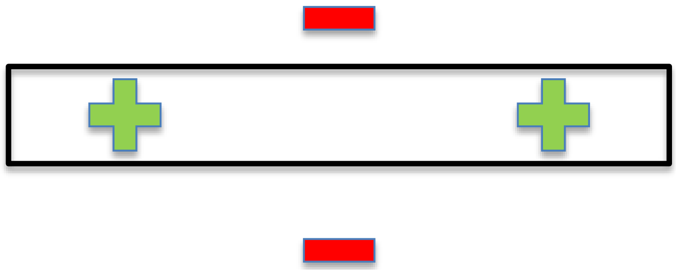
- There exists a linear separator that can shatter any set of size $d+1$ in a $d-dimensional$ space, but not $d+2$
- The larger the subset of X that can be shattered, the more expressive the hypothesis space is
- If arbitrarily large finite subsets of X can be shattered, then $VC(H)=\infty$

Axis Parallel Rectangles

- Let X be the set of all points in $\mathbb{R}^2$
- Let H be the set of all axis parallel rectangles in 2-D
  - What is $VC(H)$?

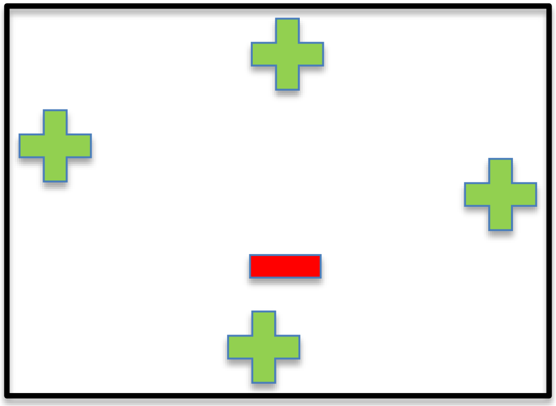
Axis Parallel Rectangles

- Let X be the set of all points in $\mathbb{R}^2$
- Let H be the set of all axis parallel rectangles in 2-D
  - $VC(H) \geq 4$



Axis Parallel Rectangles

- Let X be the set of all points in $\mathbb{R}^2$
- Let H be the set of all axis parallel rectangles in 2-D
  - $VC(H) = 4$
  - A rectangle can contain at most 4 extreme points, the fifth point must be contained within the rectangle defined by these points



PAC Bounds with VC Dimension

- VC dimension can be used to construct PAC bounds

$$m \geq \frac{1}{\epsilon}\left(4\log\frac{2}{\delta} + 8 \cdot VC(H)\log\frac{13}{\epsilon}\right)$$

- With probability at least $(1-\delta)$ every $h \in H$ satisfies

$$\epsilon_h \leq \epsilon_h^{train} + \sqrt{\frac{1}{m}\left(VC(H)\left(\ln\left(\frac{2m}{VC(H)}\right)+1\right)+\ln\frac{4}{\delta}\right)}$$

- These bounds (and the preceding discussion) only work for binary classification, but there are generalizations

PAC Learning

- Given:
  - Set of data X
  - Hypothesis space H
  - Set of target concepts C
  - Training instances from unknown probability distribution over X of the form $(x, c(x))$
- Goal:
  - Learn the target concept $c \in \mathrm{C}$

PAC Learning

- Given:
  - A concept class C over n instances from the set X
  - A learner L with hypothesis space H
  - Two constants, $\epsilon, \delta \in (0, \frac{1}{2})$
- C is said to be PAC learnable by L using H iff for all distributions over X, learner L by sampling n instances, will with probability at least $1 - \delta$ output a hypothesis $h \in \mathrm{H}$ such that
  - $\epsilon_h \leq \epsilon$
  - Running time is polynomial in $\frac{1}{\epsilon}, \frac{1}{\delta}, n, size(c)$