CS 6347

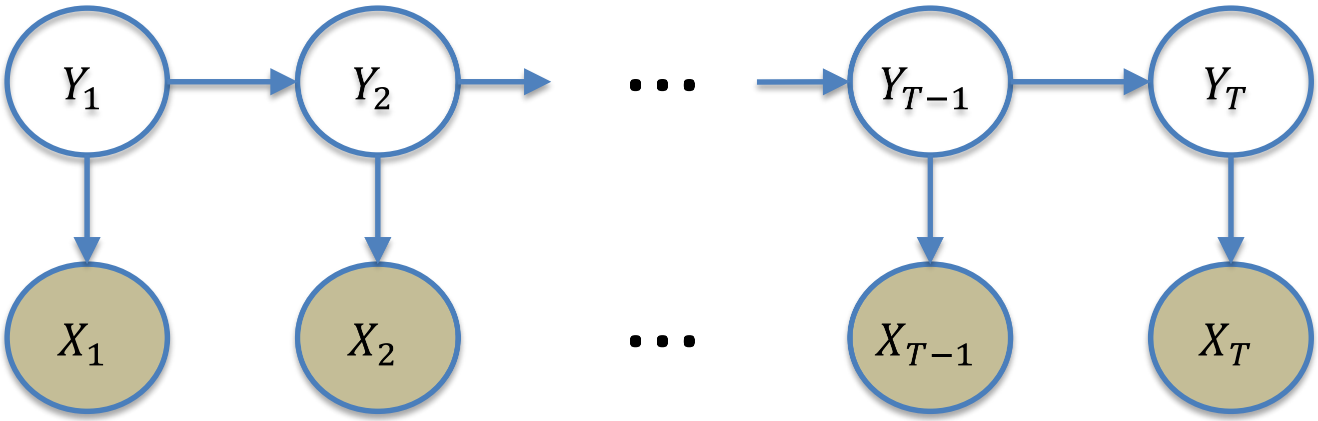
Lecture 15

Expectation Maximization

Unobserved Variables

- Latent or hidden variables in the model are never observed
  - We may or may not be interested in their values, but their existence is crucial to the model
- Some observations in a particular sample may be missing
  - Missing information on surveys or medical records (quite common)
  - We may need to model how the variables are missing

Hidden Markov Models



$$p(x_1, \ldots, x_T, y_1, \ldots, y_T) = p(y_1)p(x_1|y_1)\prod_t p(y_t|y_{t-1})p(x_t|y_t)$$

- X's are observed variables, Y's are latent
- Example: X variables correspond sizes of tree growth rings for one year, the Y variables correspond to average temperature

Missing Data

- Data can be missing from the model in many different ways
  - Missing completely at random: the probability that a data item is missing is independent of the observed data and the other missing data
  - Missing at random: the probability that a data item is missing can depend on the observed data
  - Missing not at random: the probability that a data item is missing can depend on the observed data and the other missing data

Handling Missing Data

- Discard all incomplete observations
  - Can introduce bias
- Imputation: actual values are substituted for missing values so that all of the data is fully observed
  - E.g., find the most probable assignments for the missing data and substitute them in (not possible if the model is unknown)
  - Use the sample mean/mode
- Explicitly model the missing data
  - For example, could expand the state space
  - The most sensible solution, but may be non-trivial if we don't know how/why the data is missing

Modelling Missing Data

- Add additional binary variable m_i to the model for each possible observed variable x_i that indicates whether or not that variable is observed

$$p(x_{obs}, x_{mis}, m) = p(m|x_{obs}, x_{mis})p(x_{obs}, x_{mis})$$

Modelling Missing Data

- Add additional binary variable m_i to the model for each possible observed variable x_i that indicates whether or not that variable is observed

$$p(x_{obs}, x_{mis}, m) = \underbrace{p(m|x_{obs}, x_{mis})}_{\substack{\text{Explicit model of the missing data} \\ \text{(missing not at random)}}} p(x_{obs}, x_{mis})$$

Modelling Missing Data

- Add additional binary variable m_i to the model for each possible observed variable x_i that indicates whether or not that variable is observed

$$p(x_{obs}, x_{mis}, m) = \underbrace{p(m|x_{obs})}_{\text{Missing at random}} p(x_{obs}, x_{mis})$$

Modelling Missing Data

- Add additional binary variable m_i to the model for each possible observed variable x_i that indicates whether or not that variable is observed

$$p(x_{obs}, x_{mis}, m) = \underbrace{p(m)}_{\text{Missing completely at random}} p(x_{obs}, x_{mis})$$

Modelling Missing Data

- Add additional binary variable m_i to the model for each possible observed variable x_i that indicates whether or not that variable is observed

$$p(x_{obs}, x_{mis}, m) = \underbrace{p(m)}_{\text{Missing completely at random}} p(x_{obs}, x_{mis})$$

How can you model latent variables in this framework?

Learning with Missing Data

- In order to design learning algorithms for models with missing data, we will make two assumptions
  - The data is missing at random
  - The model parameters corresponding to the missing data (δ) are separate from the model parameters of the observed data (θ)
- That is

$$p(x_{obs}, m|\theta, \delta) = p(m|x_{obs}, \delta)p(x_{obs}|\theta)$$

Learning with Missing Data

$$p(x_{obs}, m|\theta, \delta) = p(m|x_{obs}, \delta)p(x_{obs}|\theta)$$

- Under the previous assumptions, the log-likelihood of samples $(x^1, m^1), \ldots, (x^K, m^K)$ is equal to

$$l(\theta, \delta) = \sum_{k=1}^{K} \log p(m^k|x_{obs}^k, \delta) + \sum_{k=1}^{K} \log \sum_{x_{mis_k}} p(x_{obs_k}^k, x_{mis_k}|\theta)$$

Learning with Missing Data

$$p(x_{obs}, m|\theta, \delta) = p(m|x_{obs}, \delta)p(x_{obs}|\theta)$$

- Under the previous assumptions, the log-likelihood of samples $(x^1, m^1), \ldots, (x^K, m^K)$ is equal to

$$l(\theta, \delta) = \underbrace{\sum_{k=1}^{K} \log p(m^k|x_{obs}^k, \delta) + \sum_{k=1}^{K} \log \sum_{x_{mis_k}} p(x_{obs_k}^k, x_{mis_k}|\theta)}$$

Separable in θ and δ, so if we don't care about δ, then we only have to maximize the second term over θ

Learning with Missing Data

$$l(\theta) = \sum_{k=1}^{K} \log \sum_{x_{mis_k}} p(x^k_{obs_k}, x_{mis_k} | \theta)$$

- This is NOT a concave function of θ
  - In the worst case, could have a different local maximum for each possible value of the missing data
  - No longer have a closed form solution, even in the case of Bayesian networks

Expectation Maximization

- The expectation-maximization algorithm (EM) is a method to find a local maximum of the log-likelihood with missing data
- Basic idea:

$$l(\theta) = \sum_{k=1}^{K} \log \sum_{x_{mis_k}} p(x_{obs_k}^{k}, x_{mis_k} | \theta)$$

$$= \sum_{k=1}^{K} \log \sum_{x_{mis_k}} q_k(x_{mis_k}) \cdot \frac{p(x_{obs_k}^{k}, x_{mis_k} | \theta)}{q_k(x_{mis_k})}$$

$$\geq \sum_{k=1}^{K} \sum_{x_{mis_k}} q_k(x_{mis_k}) \log \frac{p(x_{obs_k}^{k}, x_{mis_k} | \theta)}{q_k(x_{mis_k})}$$

Expectation Maximization

$$F(q,\theta) \equiv \sum_{k=1}^{K} \sum_{x_{mis_k}} q_k(x_{mis_k}) \log \frac{p(x_{obs_k}^k, x_{mis_k} | \theta)}{q_k(x_{mis_k})}$$

- Maximizing F is equivalent to the maximizing the log-likelihood
- Could maximize it using coordinate ascent

$$q^{t+1} = \arg\max_{q_1,\ldots,q_K} F(q,\theta^t)$$

$$\theta^{t+1} = \operatorname{argmax}_{\theta} F(q^{t+1},\theta)$$

Expectation Maximization

$$\sum_{x_{mis_k}} q_k(x_{mis_k}) \log \frac{p(x^k_{obs_k}, x_{mis_k}|\theta)}{q_k(x_{mis_k})}$$

- This is just $-d\left(q_k || p(x^k_{obs_k}, \cdot\ |\theta)\right)$
- Maximized when $q_k(x_{mis_k}) = p(x_{mis_k}|x^k_{obs_k}, \theta)$
- Can reformulate the EM algorithm as

$$\theta^{t+1} = \underset{\theta}{\operatorname{argmax}} \sum_{k=1}^{K} \sum_{x_{mis_k}} p(x_{mis_k}|x^k_{obs_k}, \theta^t) \log p(x^k_{obs_k}, x_{mis_k}|\theta)$$

An Example: Bayesian Networks

- Recall that MLE for Bayesian networks without latent variables yielded

$$\theta_{x_i|x_{parents(i)}} = \frac{\mathrm{N}_{x_i, x_{\text{parents}(i)}}}{\sum_{x_i'} \mathrm{N}_{x_i', x_{\text{parents}(i)}}}$$

- Let's suppose that we are given observations from a Bayesian network in which one of the variables is hidden
  - What do the iterations of the EM algorithm look like?

(on board)