EXAM 2 SOLUTIONS
Spring 2003

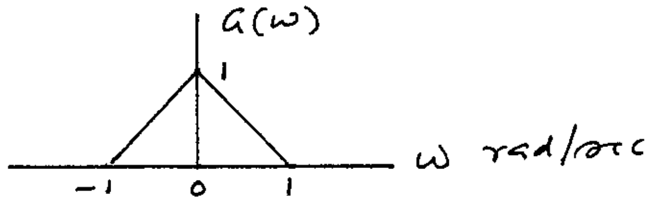
Part A

A11 (10 points)

The real signal g(t) has a spectrum G(ω) as shown in the figure below, and is sampled every T_s seconds by an ideal impulse train to generate the sampled signal $\overline{g}(t)$. *Sketch the spectrum $\overline{G}(\omega)$ of the sampled signal $\overline{g}(t)$ when $T_s = 2\pi$ seconds. Sketch the spectrum of the output signal of the reconstruction filter whose input is $\overline{g}(t)$; the frequency response of the reconstruction filter is given by*

$$H(\omega) = 2\pi\ \mathrm{rect}(\omega/2)$$

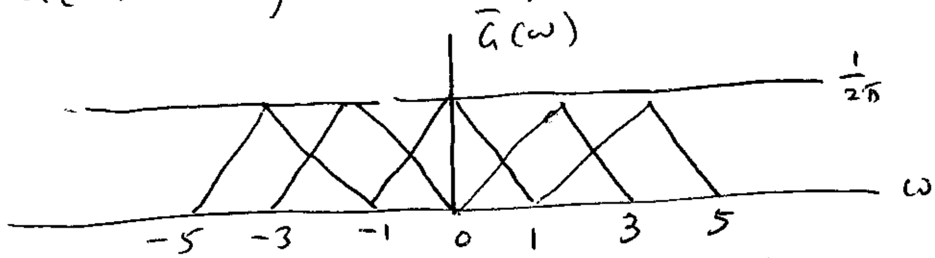
Is the output of the reconstruction filter an exact or scaled version of the signal g(t)? Give reasons.
(You must clearly mark your axes, and all the meaningful frequencies as well as amplitude levels.)

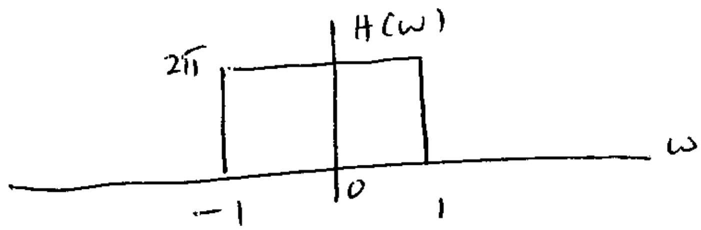


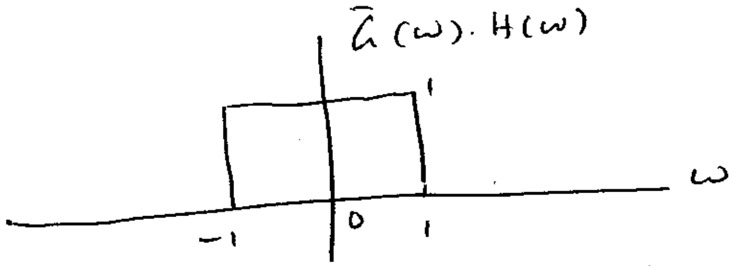
$$\omega_s = \frac{2\pi}{T_s} = \frac{2\pi}{2\pi} = 1 \text{ rad/sec}$$

$\overline{G}(\omega)$ is periodic repetition of $G(\omega)$ with periodicity ω_s.

$$\boxed{\overline{G}(\omega) = \frac{1}{2\pi}}$$







$\overline{G}(\omega)\cdot H(\omega)$ does not reconstruct the original signal due to undersampling.

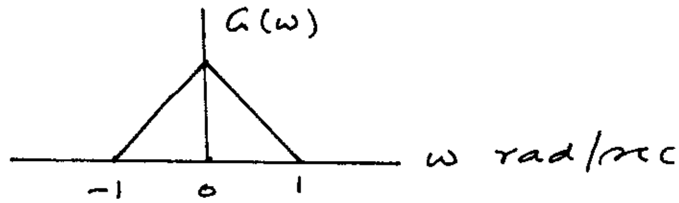
Part A

A12 (10 points)

The real signal g(t) has a spectrum G(ω) as shown in the figure below, and is sampled every T_s seconds by an ideal impulse train to generate the sampled signal $\bar{g}(t)$. *Sketch the spectrum $\bar{G}(\omega)$ of the sampled signal $\bar{g}(t)$ when $T_s = \pi$ seconds. Sketch the spectrum of the output signal of the reconstruction filter whose input is $\bar{g}(t)$; the frequency response of the reconstruction filter is given by*

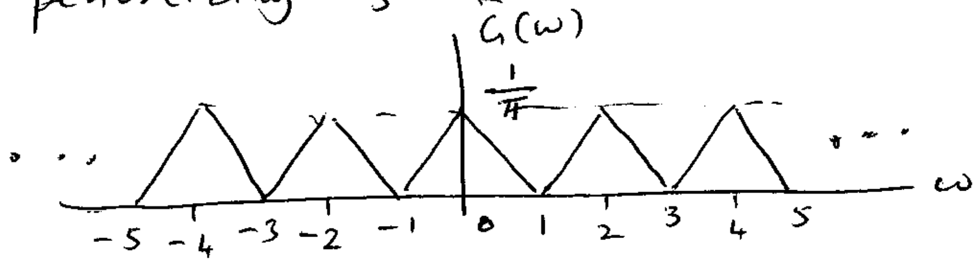
$$H(\omega) = \pi\, \mathrm{rect}(\omega/2)$$

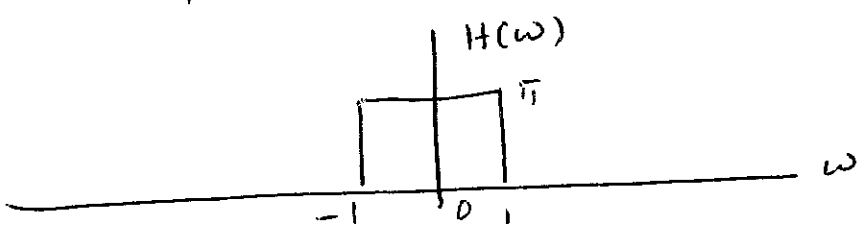
Is the output of the reconstruction filter an exact or scaled version of the signal g(t)? Give reasons.
(You must clearly mark your axes, and all the meaningful frequencies as well as amplitude levels.)

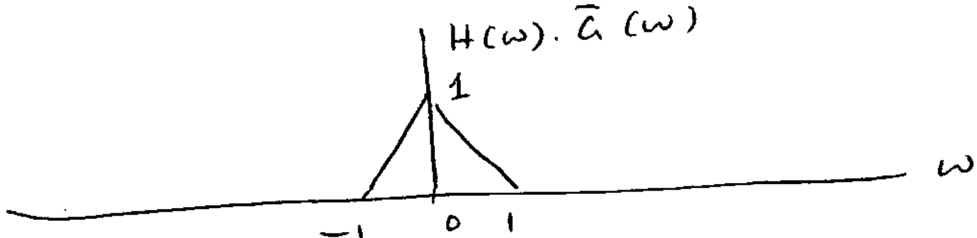


$$\omega_s = \frac{2\pi}{T_s} = \frac{2\pi}{\pi} = 2 \text{ rad/sec} \quad (\text{Nyquist rate})$$

$\bar{G}(\omega)$ is periodic repetition of G(ω) with periodicity ω_s.







$H(\omega)\bar{G}(\omega)$ is the same as G(ω). The signal g(t) is exactly reconstructed.

A21 (5 points)

A uniform quantizer operating on the samples has a data rate of 6 KBPS; the sampling rate is 1 kHz. However, the resulting signal-to-quantization noise ratio (SNR_Q) of 30 dB is unsatisfactory, and at least an SNR_Q of 40 dB is required. *What would be the minimum data rate in KBPS of the system that meets the requirement? What would be the minimum transmission bandwidth required if 4-ary signaling is used?* Show all your steps/logic as necessary.

$DR_{BPS} = 6$ KBPS $f_s = 1$ kHz $\Rightarrow B_n = 6$ bits/sample

but $SNR_Q = 30$ dB is unacceptable. To increase to $SNR_Q \geq 40$ dB, we need to increase $B_n' = 8$ bits (6 dB/bit). The new data rate $DR_{BPS} = f_s \cdot B_n' = \boxed{8 \text{ KBPS}}$

$$B_T \geq \frac{1}{2} \cdot \frac{DR_{BPS}}{\log_2 M} \Rightarrow \boxed{B_T \geq 2 \text{ kHz}} \quad \boxed{\text{Min BW} = 2 \text{kHz}}$$

A31 (5 points)

A signal m(t) has a bandwidth of 2 kHz and exhibits a maximum rate of change of 2 volt/second. The signal is sampled at five times the Nyquist rate and quantized using a delta modulator. What should be the minimum step size to avoid slope overload?

$f_s = 5 \times 2 \times 2$ kHz $= 20$ kHz $\quad T_s = \frac{1}{f_s} = \frac{1}{2} 10^{-4}$ sec

$$\left|\frac{dm(t)}{dt}\right|_{max} = 2 \text{ volts/sec}$$

For slope overload avoidance, we require:

$$\frac{\delta}{T_s} \geq \left|\frac{dm(t)}{dt}\right|_{max}$$

$$\frac{\delta}{1/2 \times 10^{-4}} \geq 2 \Rightarrow \boxed{\delta \geq 10^{-4} \text{ volts}}$$

$$\boxed{\delta_{min} = 0.1 \text{ mV}}$$

A22 (5 points)

A signal of bandwidth 1 kHz is sampled at 50% over the Nyquist rate, and each sampled is quantized to 8 bits. *What is the data rate in KBPS? If 8-ary signaling is used to transmit the data, what is the minimum required transmission bandwidth in kHz?* Show all your steps/logic as necessary.

$f_s = 1.5 \times 2 \times 1\ kHz = 3\ kHz \qquad B_n = 8$ bits/samp

$DR_{BPS} = 3 \times 10^3 \times 8 = \boxed{24\ KBPS}$

$M = 8.$

$$B_T \geq \frac{1}{2} \frac{DR_{BPS}}{\log_2(M)} \Rightarrow B_T \geq \frac{1}{2} \frac{24}{3}\ kHz$$

$$\Rightarrow \boxed{B_T \geq 4\ kHz}$$

Min Trans. BW = $\boxed{4\ kHz}$

A32 (5 points)

A PCM system, at 10 KBPS with 10 bits/sample, has a *satisfactory* SNR_Q of 50 dB. *Binary signaling is required* and the available transmission bandwidth is 3 kHz. A DPCM system has to be used to transmit the signal in the bandwidth specified. *What is the required prediction gain (in dB) if the* SNR_Q *of 50 dB is to be maintained?* Your should show all relevant logic in arriving at the answer.

Through a channel of BW 3 kHz, with binary signaling we can only transmit a max. of 6 KBPS. At $f_s = 1$ kHz (from the given data), we must have $b_n = 6$ bits/sampl. However, the PCM system is operating at $b_n = 10$ bits/sampl. To maintain the same SNR_Q, the DPCM prediction gain must correspond to a "4-bit improvement", i.e. the prediction gain must be at least 4×6 dB =

$\boxed{24\ dB}$.

A41 (5 points)

Let x(t) be a periodic square wave with a fundamental period of 1 millisecond. Is it possible to sample this signal and reconstruct it exactly from the sampled signal? Explain your answer.

x(t) has infinite bandwidth, i.e. it is not bandlimited. It is not possible to sample and reconstruct x(t).

A51 (10 points)

A signal m(t) = 2 cos ($2\pi\, 10^3$ t) frequency modulates (FM) a 1 MHz carrier to produce a (peak) frequency deviation of 4 kHz. *Write the time-domain expression for the resulting FM signal $\phi_{FM}(t)$? What is the value of the frequency deviation constant k_f? What is the bandwidth of the FM signal?*

$$\beta = \frac{\Delta f}{f_m} = \frac{4 \times 10^3}{10^3} = 4.$$

$$\boxed{\phi_{FM}(t) = A \cdot \cos\left(2\pi \cdot 10^6 t + 4 \cdot \sin 2\pi 10^3 t\right)}$$

$$\Delta f = \frac{1}{2\pi} \cdot k_f \cdot m_p$$

$$\cancel{4}^{2} \times 10^3 = \frac{1}{2\pi} \cdot k_f \cancel{2} \Rightarrow \boxed{k_f = 4\pi \times 10^3 \text{ rad/sec/volt}}$$

$$BW_{FM} = 2(\Delta f + BW)$$

$$= 2(4+1)\text{ kHz} = \boxed{10 \text{ kHz}}$$

A42 (5 points)

What are the various conditions that result in the aliasing occur in the sampling of signals?

- undersampling, i.e. $f_s < 2\,BW$ for bandlimited signal
- signal is not bandlimited.

A52 (10 points)

A signal $m(t) = 2\cos(2\pi\, 10^3\, t)$ phase modulates (PM) a 1 MHz carrier to produce a (peak) frequency deviation of 4 kHz. *Write the time-domain expression for the resulting PM signal $\phi_{PM}(t)$? What is the value of the phase deviation constant k_p? What is the bandwidth of the PM signal?*

$$\Delta f = \frac{1}{2\pi} \cdot k_p \cdot \left|\frac{dm}{dt}\right|_{max}$$

$$\cancel{4} \times \cancel{10^3} = \frac{1}{2\cancel{\pi}} k_p \cdot (\cancel{4\pi}\, \cancel{10^3}) \Rightarrow \boxed{k_p = 2 \text{ rad/volt}}$$

$$\boxed{\phi_{PM}(t) = A\cos\left(2\pi 10^6 t + 4\cos 2\pi \cdot 10^3 t\right)}$$

$$BW_{PM} = 2(\Delta f + BW) = 2(4+1)\text{ kHz}$$

$$\boxed{BW_{PM} = 10\text{ kHz}}$$

Part B (40 points)

B11 (20 points)

An angle modulated signal is given by the following expression:

$\psi(t)$

$$\varphi_{EM}(t) = 5\cos(\omega_c t + \underbrace{40\sin 500\pi t + 20\sin 1000\pi t + 10\sin 2000\pi t}) \quad \text{—(1)}$$

a. Determine the (peak) frequency deviation Δf, in Hz.
b. Estimate the bandwidth, in Hz, of the angle modulated signal by Carson's rule.
c. .If the angle modulated signal is a phase modulated signal with the phase deviation constant, k_p is 5 radians per volt, determine the message signal m (t).
d. If the angle modulated signal is a frequency modulated signal with a frequency deviation constant, k_f is 20,000 π radians/sec per volt, determine the message signal m(t).

a.
$$\omega_{i_{max}} - \omega_c = \left| \frac{d}{dt}\left[40\sin 500\pi t + 20\sin 1000 t + 10\sin 2000 t\right]\right|_{max}$$
$$= \left|20{,}000\pi\cos 500\pi t + 20{,}000\pi\cos 1000 t + 20{,}000\pi\cos 2000 t\right|$$
$$= 60{,}000\pi \text{ rad/sec}$$
$$\Rightarrow \boxed{\Delta f_{EM} = 30 \text{ kHz}}$$

b.
$$BW_{EM} = 2(\Delta f_{EM} + BW) = 2(30+1)\,\text{kHz} = \boxed{62 \text{ kHz}}$$

c.
$$k_p = 5 \text{ rad/volt}$$
$$\phi_{PM}(t) = A\cos(\omega_c t + k_p \cdot m(t)) \quad \text{—(2)}$$
Making correspondences between (1) & (2), we have
$$\boxed{m(t) = 8\sin 500\pi t + 4\sin 1000 t + 2\sin 2000 t}$$

d
$$k_f = 20{,}000\pi \text{ rad/sec/volt}$$
$$\omega_i(t) = k_f \cdot \frac{d\psi(t)}{dt} \quad \text{—(3)}$$
3.4
$$= k_f\, m(t)$$
$$\boxed{m(t) = \cos 500\pi t + \cos 1000\pi t + \cos 2000\pi t}$$

Part B (40 points)

B12 (20 points)

A message signal m(t) = 4 cos 2π 1000t modulates a carrier frequency to produce a frequency modulated signal with a resulting modulation index (i.e. frequency deviation ratio) of 2.

(a) *What is the estimate of the bandwidth of the FM signal ?*

(b) *The message signal m(t) is replaced by a new message signal*
m(t) = 4 cos 2π 1000t + 4 cos 2π 3000t
What is the estimate of the bandwidth of this new FM signal?

(c) *The message signal m(t) is replaced by a new message signal*
m(t) = 4 cos 2π 3000t
What is the estimate of the bandwidth of this new FM signal?

Use Carson's rule in determining the bandwidth of FM signals.

FM

a. $\beta = 2, \quad f_m = 1\,KHz \qquad \Delta f = \beta f_m = 2\,kHz.$

$BW_{FM} = 2(\Delta f + BW) = 2(2+1)\,kHz = \boxed{6\,kHz}$

b. From a, $\Delta f = 2\times10^3 = \frac{1}{2\pi}\cdot k_f \cdot m_p$

with $m_p = 4$ from part a $\Rightarrow k_f = \frac{2\pi\cdot\Delta f}{m_p} = \frac{\cancel{2}\pi \times \cancel{2}\times 10^3}{\cancel{4}}$

$= 10^3\pi$ rad/sec/volt

The signal is changed to

$m(t) = 4\cos 2\pi 1000t + 4\cos 2\pi 3000t$

with new $m_p = 8$, but k_f is unchanged.

$\Delta f_{new} = \frac{1}{2\pi}\, m_{p_{new}} \cdot k_f = \frac{1}{\cancel{2\pi}}\cdot \overset{4}{\cancel{8}}\cdot\cancel{\pi}\times 10^3 = \boxed{4\,kHz}$

for m(t) $\rightarrow$ BW = 3 kHz.

$BW_{FM} = 2(\Delta f + BW)$
$= 2(4+3) = \boxed{14\,kHz}$

$\boxed{BW_{FM} = 14\,kHz}$

c. $k_f = \pi\cdot10^3$ rad/sec/volt (unchanged), but

$m(t) = 4\cos 2\pi 3000t \Rightarrow m_{p_{new}} = 4$, BW = 3 kHz

$\boxed{\Delta f = 2\,kHz}$

$BW_{FM} = 2(\Delta f + BW) = 2(2+3)\,kHz$

$\boxed{BW_{FM} = 10\,kHz}$

B21 (20 points)

Two signals, $m_1(t)$ and $m_2(t)$, are to be sampled, quantized, and multiplexed for transmission by *binary signaling*. Each of the signals has a bandwidth of 2 kHz. The samples of the signal $m_1(t)$ is uniformly distributed between –1.5 volts and 1.5 volts. The RMS value of the samples of the signal $m_2(t)$ is one tenth the peak amplitude. Both signals are sampled at *twice the Nyquist rate. Each of the signals must have a minimum SNR_Q of 46 dB.* Three types of quantizers are available: (1) uniform quantizer, (2) μ-law quantizer with μ= 255 and (3) A-law with A=87.6. *Zero overhead* is assumed for framing and synchronization. *Choose appropriate quantizer for each signal to minimize the data rate and determine the minimum transmission bandwidth in kHz.*

m_1: $f_{s_1} = 2 \times 2 \times 2$ kHz $= 8$ kHz

m_2: $f_{s_2} = 2 \times 2 \times 2$ kHz $= 8$ kHz

m_1: uniform distribution $\Rightarrow \frac{m_p^2}{\overline{m^2}} = 3$.

Uniform Quant.

$$SNR_{Q_{uniform}} = 6.02 \cdot B_n + 4.77 - \cancel{10\log_{10} 3} \geq 46 \text{ dB}$$

$$\Rightarrow \boxed{B_n = 8 \text{ bits}}$$

μ-Law, μ = 256

$$SNR_{Q_\mu} = \left[6.02 B_n + 4.77 - \overbrace{20\log_{10}(\ln 256)}^{14.88}\right] \geq 46 \text{ dB}$$

$$\Rightarrow \boxed{B_n = 10 \text{ bits}}$$

A-Law, A = 87.6

$$SNR_{Q_A} = \left[6.02 B_n + 4.77 - \overbrace{20\log_{10}[1 + \ln 87.6]}^{14.76}\right] \geq 46 \text{ dB}$$

$$\Rightarrow \boxed{B_n = 10 \text{ bits}}$$

$\boxed{\text{For } m_1, \text{ choose uniform quantizer, } B_n = 8 \text{ bits}}$

m_2: $\frac{m_p}{m_{rms}} = 10 \Rightarrow \frac{m_p^2}{\overline{m^2}} = 100$

uniform quant: $SNR_Q = \left[6.02 B_n + 4.77 - \overbrace{10\log_{10} 100}^{20}\right] \geq 46 \text{ dB}$

$$\Rightarrow \boxed{B_n = 11 \text{ bits}}$$

For μ-law or A-law, we need $\boxed{B_n = 10 \text{ bits}}$

$\boxed{\text{For } m_2, \text{ choose A-law or μ-Law, } B_n = 10 \text{ bits}}$

B21 (continued)

Raw ~~Data~~ Rate:

$$DR_{BPS} = f_{s_1} B_{n_1} + f_{s_2} B_{n_2}$$

$$= 8 \times 10^3 \times 8 + 8 \times 10^3 \times 10 = \boxed{144 \text{ KBPS}}$$

$\boxed{\text{Minimum Data Rate: } 144 \text{ KBPS}}$

(zero overhead)

Binary Signaling $\Rightarrow$

$$B_T \geq \frac{1}{2} DR_{BPS}$$

$$B_T \geq \frac{1}{2} \cdot 144 \text{ KHz} \Rightarrow \boxed{B_T \geq 72 \text{ KHz}}$$

$\boxed{\text{Min. Trans. BW} = 72 \text{ KHz}}$

B22 (20 points)

Two signals, $m_1(t)$ and $m_2(t)$, are to be sampled, quantized, and multiplexed for transmission by *binary signaling*. Each of the signals has a bandwidth of 4 kHz. The samples of the signal $m_1(t)$ is uniformly distributed between −2.5 volts and 2.5 volts. The RMS value of the samples of the signal $m_2(t)$ is one sixteenth the peak amplitude. Both signals are sampled at *twice the Nyquist rate. Each of the signals must have a minimum SNR_Q of 46 dB.* Three types of quantizers are available: (1) uniform quantizer, (2) μ-law quantizer with μ= 255 and (3) A-law with A=87.6. *Zero overhead* is assumed for framing and synchronization. *Choose appropriate quantizer for each signal to minimize the data rate and determine the minimum transmission bandwidth in kHz.*

m_1 & m_2 are sampled at $f_{s_1} = f_{s_2} = 2\times2\times4 = \boxed{16\text{ kHz}}$

m_1: uniform distribution $\Rightarrow \dfrac{m_p^2}{\overline{m^2}} = 3$

uniform Quant.: $SNR_Q = \left[6.02 B_n + \cancel{4.77 - 10\log_{10}3}\right] \geq 46\text{ dB}$

$\Rightarrow \boxed{B_n = 8\text{ bits}}$

μ-Law, μ=255: $SNR_Q = \left[6.02 B_n + 4.77 - \underbrace{20\log_{10}[\ln 256]}_{14.88}\right] \geq 46\text{ dB}$

$\Rightarrow \boxed{B_n = 10\text{ bits}}$

A-Law, A=87.6: $SNR_Q = \left[6.02 B_n + 4.77 - \overbrace{20\log_{10}[1+\ln 87.6]}^{14.76}\right] \geq 46\text{ dB}$

$\Rightarrow \boxed{B_n = 10\text{ bits}}$

$\boxed{\text{For } m_1\text{, choose uniform quantizer with } B_n = 8\text{ bits.}}$

m_2: $\dfrac{m_p}{m_{rms}} = 16 \Rightarrow \dfrac{m_p^2}{\overline{m^2}} = 256$

uniform quant.: $SNR_Q = \left[6.02 B_n + 4.77 - \overbrace{10\log_{10}256}^{24.1}\right] \geq 46\text{ dB}$

$\Rightarrow \boxed{B_n = 11\text{ bits}}$

For μ-Law & A-law, we have $B_n = 10$ bits

$\boxed{\text{For } m_2\text{, choose μ-Law or A-law with } B_n = 10\text{ bits}}$

B22 (continued)

Data Rate (assuming zero overhead):

$$DR_{BPS} = f_{s_1} B_{n_1} + f_{s_2} B_{n_2}$$

$$= 16 \times 10^3 \times 8 + 16 \times 10^3 \times 10 = \boxed{288 \text{ kBPS}}$$

$\boxed{\text{Min. Data Rate} = 288 \text{ KBPS}}$

<u>Trans. BW:</u>

(Binary Signaling)

$$B_T \geq \frac{1}{2} DR_{BPS}$$

$$B_T \geq \frac{1}{2} \cdot 288 \text{ kHz}$$

$$\Rightarrow \boxed{B_T \geq 144 \text{ kHz}}$$

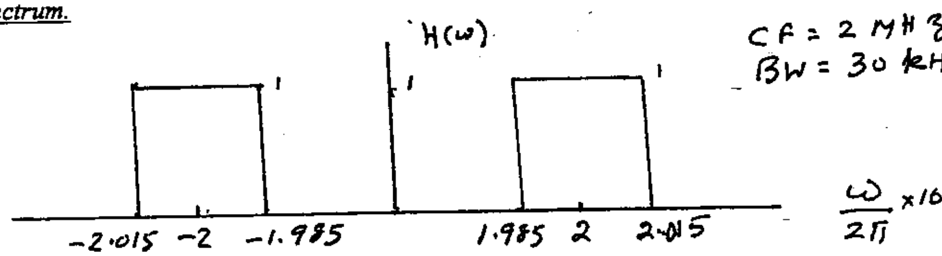
Min. Trans. BW = 144 kHz

C11 (20 points)

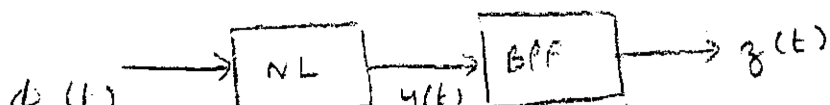
An FM signal

$$\varphi_{FM}(t) = 5\cos(2\pi.10^6 t + 2.5\sin 20{,}000\pi t)$$

is input to a square-law nonlinearity (with the characteristic: $y = 2x^2$, where x is the input and y is the output), and filtered by a bandpass filter to produce the output z(t). The frequency response of the bandpass filter is shown below. *Determine the output z(t), and sketch its* ***magnitude spectrum*** *and* ***phase spectrum.***



CF = 2 MHz
BW = 30 kHz



$$y(t) = 2\varphi_{FM}^2(t) = 2\left[A\cos(\omega_c t + \psi(t))\right]^2$$

$$= 2\left[\frac{A^2}{2} + \frac{A^2}{2}\cos(2\omega_c t + 2\psi(t))\right]$$

$$= \underset{\uparrow\ \text{Rejected by BPF}}{A^2} + \underbrace{A^2\cos(2\omega_c t + 2\psi(t))}_{\to\, y_1(t)}$$

$$y_1(t) = A^2\cos(2\omega_c t + 2\psi(t))$$

For our case, $A = 5$, $\omega_c = 2\pi.10^6$ and $\psi(t) = 2.5\sin 20{,}000\pi t$

$$y_1(t) = 25\cos(2\pi.2.10^6 t + 5\sin 20{,}000\pi t) \quad \text{—(1)}$$

$y_1(t)$ in (1) is an FM signal with $\beta = 5$ and modulating signal tone of 10 kHz. (1) can be expanded into a series

$$y_1(t) = 25\sum_{n=-\infty}^{\infty} J_n(5)\cos(2\pi(2.10^6 t + n.10.10^3 t)) \quad \text{—(2)}$$

C11 (Continued)

The bandpass filter will pass only the following frequencies

- Carrier at 2 MHz
- First upper sideband at 2.01 MHz
- First lower sideband at 1.99 MHz

All other frequencies in ② will be rejected with the given BPF.

Therefore, we can write

$$z(t) = 25\left[J_{-1}(5)\cos(2\pi\cdot(1.99)\cdot 10^6 t) + J_0(5)\cos(2\pi\cdot 2\cdot 10^6 t) + J_1(5)\cos(2\pi\,(2.01)10^6 t)\right] \quad -③$$

where

$$J_{-1}(5) = (-1)^1 J_1(5) = -(-0.328) = 0.328$$

$$J_0(5) = -0.178$$

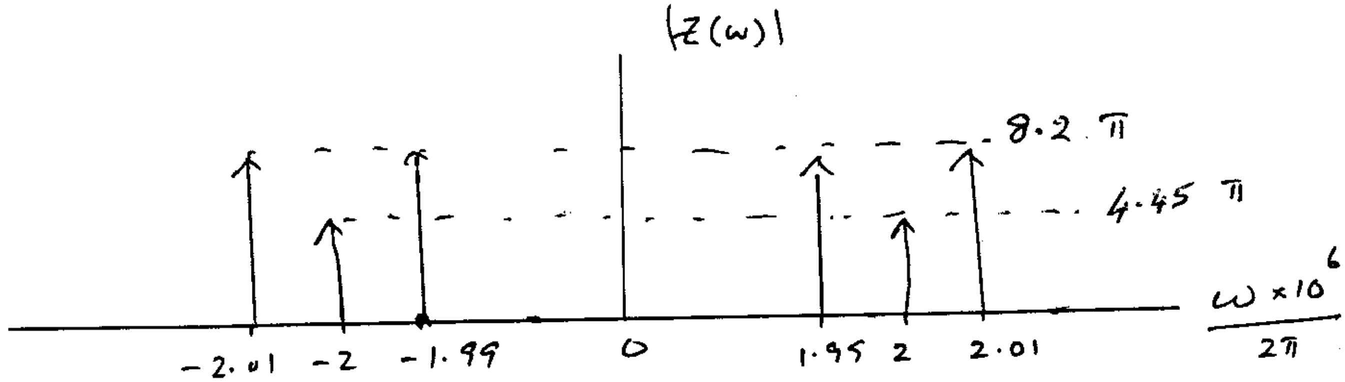
and $J_1(5) = -0.328$

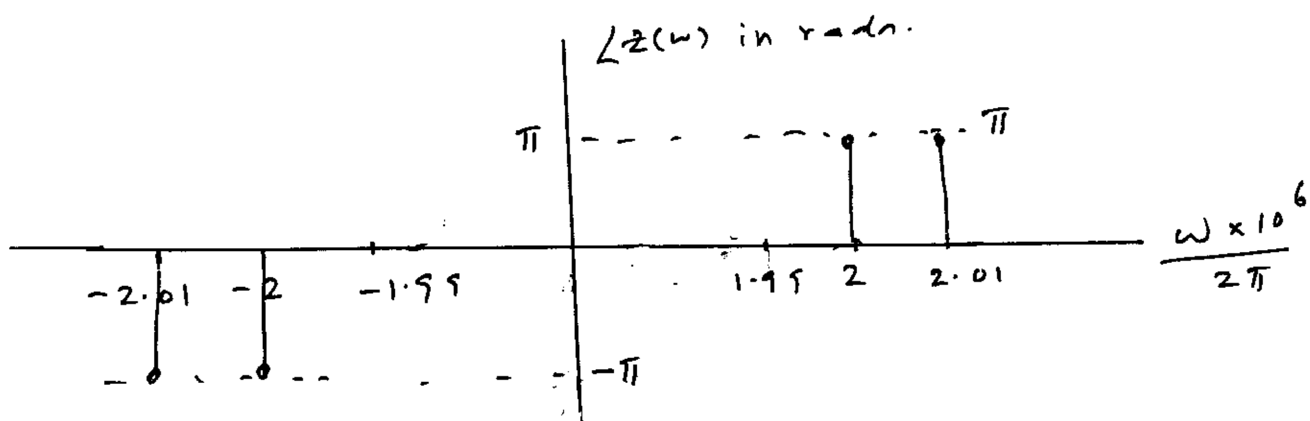
Thus, ③ can be written as:

$$\boxed{z(t) = \left[(8.2)\cos(2\pi\,(1.99)10^6 t) - (4.45)\cos(2\pi\, 2\cdot 10^6 t) - (8.2)\cos(2\pi\,(2.01)10^6 t)\right]}$$

$$z(t) = (8.2)\cos\left(2\pi(1.99)10^6 t\right)$$
$$+ (4.45)\cos\left(2\pi \cdot 2\, 10^6 t\right) \cdot e^{j\pi}$$
$$+ (8.2)\cos\left(2\pi(2.01)10^6 t\right) e^{j\pi}$$

The magnitude and phase spectra are shown below





Note: The phase spectrum shown above could be inverted, and that result is also correct.

C12 (20 points)

A signal m(t) frequency modulates a 100 kHz carrier to produce the following narrowband FM signal:

$$\varphi_{NBFM}(t) = 5\cos(2\pi.10^5 t + 0.025\sin 2\pi\ 10^3 t).$$

Generate (block diagram design) the wideband FM signal $\varphi_{WBFM}(t)$ with a carrier frequency of 100 MHz and a (peak) frequency deviation of 75 kHz. Assume that the following are available for the design:

1. Frequency Multipliers of any (integer) value
2. A local oscillator whose frequency can be tuned to any value between 120 MHz to 500 MHz
3. An ideal bandpass filter with tunable center frequency and bandwidth.

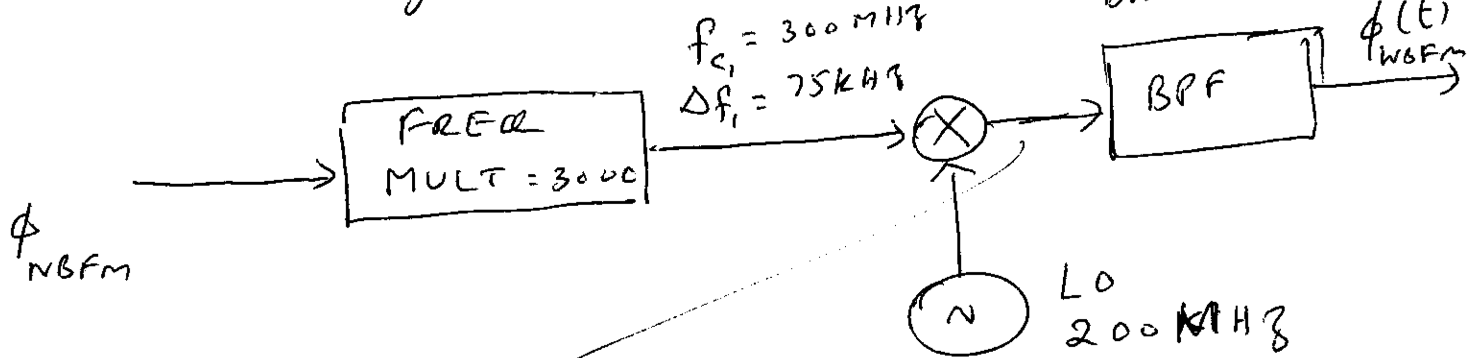
Your block diagram design must clearly specify the carrier frequencies and frequency deviations at all logical points, as well as the center frequency and bandwidth of the bandpass filter.

$$\Delta f_0 = \beta \cdot f_m = 0.025 \times 10^3 = 25\ Hz$$

$$\frac{\Delta f}{\Delta f_0} = \frac{75\ kHz}{25} = \boxed{3000} \qquad \frac{f_c}{f_0} = \frac{100\ MHz}{100\ kHz} = \boxed{1000}$$

"Practically No Constraint!"

CF = 100 MHz
BW = 152 kHz



$\Delta f_2 = 75$ kHz

$f_{c_2} = 100$ MHz

$f'_{c_2} = 500$ MHz $\longrightarrow$ rejected by BPF.