The Simplex Method: An Example

Our first step is to introduce one more new variable, which we denote by z. The variable z is *define* to be equal to $4x_1 + 3x_2$. Doing this will allow us to have a unified statement of the objective function, namely: Maximize z. Of course, we must at the same time introduce $z = 4x_1 + 3x_2$ into the constraint set, so that the new variable z is properly linked to the other variables. Following our convention of putting all variables on the left-hand side of the constraints, we will write this new constraint as: $z - 4x_1 - 3x_2 = 0$. With these revisions, we now have the following equivalent problem:

$$
\begin{array}{lrrrrrrrrrl}
\text{Maximize} & z & & & & & & & & & \\
\text{Subject to:} & & & & & & & & & & \\
& z & -4x_1 & -3x_2 & & & & & = & 0 & (0) \\
& & 2x_1 & +3x_2 & +s_1 & & & & = & 6 & (1) \\
& & -3x_1 & +2x_2 & & +s_2 & & & = & 3 & (2) \\
& & & 2x_2 & & & +s_3 & & = & 5 & (3) \\
& & 2x_1 & +x_2 & & & & +s_4 & = & 4 & (4)
\end{array}
$$

$$x_1,\ x_2,\ s_1,\ s_2,\ s_3,\ s_4 \geq 0\,,$$

where we have labelled the new constraint as equation (0). Note that the new variable z is unrestricted in sign.

More generally, it can be shown that any given linear program can be converted into the above format, which will henceforth be referred to as the *standard form*. The standard form provides a unified starting configuration for the solution of a linear program by the Simplex method. We will return to a further discussion on how to convert problems into the standard form later.

Our next step is to construct an initial basic feasible solution based on the configuration of equations (1)–(4). Clearly, with equation (0) excluded, we have a set of 4 functional equality constraints in 6 unknowns. Therefore (from our discussion in the previous section), each basic feasible solution will have two nonbasic variables, whose values are set to zero, and four basic variables, whose (nonnegative) values are determined (uniquely) by solving four equations in the four remaining unknowns. In principle, any particular choice of a pair of "candidate" nonbasic variables can *potentially* lead to a basic feasible solution, and this will indeed be the case if the values in the solution of the resulting four equations turn out to be all nonnegative. Since, in general, this outcome cannot be guaranteed a priori, the question is: Which pair should we choose? To answer this question, we will take a close look at equations (1)–(4). Notice that each of these four equations contains a variable that appears in that equation only. These variables are s_1, s_2, s_3, and s_4. Now, if we designate x_1 and x_2 as nonbasic and set their values to zero, then the resulting four equations assume the following most simple form:

$$\begin{array}{rrrrcl} +s_1 & & & & = & 6 \\ & +s_2 & & & = & 3 \\ & & +s_3 & & = & 5 \\ & & & +s_4 & = & 4\,; \end{array}$$

and hence, no effort is required in solving these equations. Moreover, the fact that all four right-hand side constants are nonnegative implies that the resulting augmented solution, $(x_1,\, x_2,\, s_1,\, s_2,\, s_3,\, s_4) = (0,\, 0,\, 6,\, 3,\, 5,\, 4)$, is necessarily feasible. Thus, the particular configuration of equations (1)–(4) allows us to "conveniently" generate (or read out) a basic feasible solution, namely (0, 0, 6, 3, 5, 4). We shall say that (0, 0, 6, 3, 5, 4) is the basic feasible solution "associated" with equations (1)–(4), and we will also let it be our starting basic feasible solution. Note that this solution corresponds to point A in the graph.

To determine the value of z associated with this starting basic feasible solution, we turn to equation (0). Note that the variable z appears only in equation (0), and that none of the current basic variables, s_1, s_2, s_3, s_4, appears in that equation. Therefore, with both x_1 and x_2 assigned the value zero, z must equal to the constant sitting at the right-hand side of equation (0). This shows that the objective-function value associated with (0, 0, 6, 3, 5, 4) is equal to 0.

We shall refer to the set of basic variables in a basic solution as the *basis*. Thus, the variables s_1, s_2, s_3, and s_4 constitute the basis in the solution $(x_1,\, x_2,\, s_1,\, s_2,\, s_3,\, s_4) = (0,\, 0,\, 6,\, 3,\, 5,\, 4)$. Note that every constraint equation has exactly one associated basic variable.

The next question is whether or not the current basic feasible solution, or point A, is optimal. To answer this, it is helpful to imagine yourself standing at point A and attempt to travel toward the direction of either point B or point E, along the x_2 axis or the x_1 axis, respectively. In the discussion below, we will switch freely between the graphical viewpoint and a corresponding "algebraic development." The graphical viewpoint helps us understand the algebraic development, but in the end, only the latter constitute the final solution procedure.

Consider point B first. Algebraically, "travelling along the x_2 axis" means that we will attempt to increase x_2 from its current value 0, while holding the value of x_1 at 0. We will "emulate" this action parametrically, as follows. Suppose x_2 is increased by a nominal amount, say δ. Notice that, in order to maintain feasibility while making such a move, we must also readjust the values of the current basic variables, s_1, s_2, s_3, and s_4, appropriately. Specifically, maintaining feasibility means that we must dynamically (or continuously) enforce the equalities in

$$\begin{array}{rrrrrrrr} 2\times 0 & +3\times(0+\delta) & +s_1 & & & & = & 6 \\ -3\times 0 & +2\times(0+\delta) & & +s_2 & & & = & 3 \\ & 2\times(0+\delta) & & & +s_3 & & = & 5 \\ 2\times 0 & +(0+\delta) & & & & +s_4 & = & 4 \end{array}$$

as we increase δ. In the first equation here, s_1 was equal to 6; but as a result of the new contribution 3δ from the nominal increase in x_2 (from 0 to $0+\delta$), the value of s_1 must be adjusted downward by the same amount to offset this new contribution. That is, s_1 must assume the new value $6-3\delta$. Similar reasoning shows that the other current basic variables s_2, s_3, and s_4 must assume the new values $3-2\delta$, $5-2\delta$, and $4-\delta$, respectively. This results in the new augmented solution $(x_1, x_2, s_1, s_2, s_3, s_4) = (0, \delta, 6-3\delta, 3-2\delta, 5-2\delta, 4-\delta)$. Now, since all variables must remain nonnegative, we also need to require that $6-3\delta \geq 0$, $3-2\delta \geq 0$, $5-2\delta \geq 0$, and $4-\delta \geq 0$. This implies that the value of δ should not exceed

$$\min\left[\frac{6}{3}, \frac{3}{2}, \frac{5}{2}, \frac{4}{1}\right] = \frac{3}{2}$$

("min" means "the minimum of"). Note that, with $\delta = 3/2$, the new augmented solution becomes $(x_1, x_2, s_1, s_2, s_3, s_4) = (0, 3/2, 3/2, 0, 2, 5/2)$; and this is precisely point B.

What we have shown is that by increasing the value of δ from 0 to 3/2, the family of augmented solutions $(0, \delta, 6-3\delta, 3-2\delta, 5-2\delta, 4-\delta)$ "trace out" all points on the AB edge of the feasible region. Note that with $\delta = 3/2$, the value of s_2 is reduced to 0, which means that the second original inequality constraint becomes binding. Thus, if we allow δ to go beyond 3/2, then the resulting augmented solutions will be infeasible.

To determine whether or not point B is better than point A, observe that from equation (0), we have

$$z \quad -4\times 0 \quad -3\times(0+\delta) \quad = \quad 0\,.$$

Hence, with $\delta = 3/2$, the value of z will be revised from 0 to $3\delta = 9/2$, a better value. Note in addition that, in response to an increase in δ, the value of z goes up at a rate of 3.

The analysis for point E is similar. Consider a nominal increase of size δ for x_1. Since the value of x_2 will be held at 0, we see from constraints (1)–(4) that the corresponding values for s_1 s_2, s_3, and s_4 should be revised to: $6-2\delta$, $3+3\delta$, $5-0\times\delta$, and $4-2\delta$, respectively. Thus, the new augmented solution is $(x_1, x_2, s_1, s_2, s_3, s_4) = (\delta, 0, 6-2\delta, 3+3\delta, 5, 4-2\delta)$. Again, the nonnegativity requirements imply that δ should not exceed

$$\min\left[\frac{6}{2}, \frac{4}{2}\right] = 2$$

(s_2 and s_3 cannot be made negative by increasing δ). With $\delta = 2$, the new augmented solution becomes $(x_1, x_2, s_1, s_2, s_3, s_4) = (2, 0, 2, 9, 5, 0)$; and this corresponds to point E. Finally, from equation (0), we have

$$z \quad -4 \times (0 + \delta) \quad -3 \times 0 \quad = \quad 0\,.$$

Hence, with $\delta = 2$, the value of z will be revised from 0 to $4 \times \delta = 8$, which again is a better value. In addition, note that the value of z increased at a rate of 4.

The above analysis shows that point A is not optimal, and we can travel to either point B or point E to achieve a better objective-function value. We will select point E, since it offers a greater improvement (9/2 for B versus 8 for E). This choice, in fact, also offers a greater *rate* of improvement in the value of z (3 for B versus 4 for E).

Next, we will focus our attention on point E, and the immediate question again is: Is E optimal?

The above analysis for point A can be viewed as an *optimality test*, which we plan to duplicate for point E. Before proceeding, it is however important to realize that the optimality test for point A was greatly facilitated by the particular algebraic configuration of the constraints. Specifically, recall that the choice of point A as the initial basic feasible solution was motivated by the following three convenient attributes of the standard form:

1. Each of constraints (1)–(4) contains a variable that has a coefficient of 1 and appears in that equation only. These variables, s_1, s_2, s_3, and s_4, serve as candidate basic variables.

2. The constants on the right-hand side of equations (1)–(4) are all nonnegative.

3. The candidate basic variables do not appear in constraint (0).

Therefore, it seems natural to attempt to recast the original constraint set into an equivalent standard form that is better suited for an optimality test for point E. This will be our next task.

Observe that the augmented solution that corresponds to point E, namely (2, 0, 2, 9, 5, 0), assigns a zero to both x_2 and s_4. This means that at point E, the nonbasic variables are x_2 and s_4, and the basic variables are x_1, s_1, s_2, and s_3. Thus, switching from point A to point E can be reinterpreted as changing (i) the status of x_1 from being nonbasic to being basic and (ii) the status of s_4 from being basic to being nonbasic. In other words, as we switch from A to E, the variable x_1 is to enter the basis and the variable s_4 is to leave the basis. Now, with x_1, s_1, s_2, and s_3 being the new basis, what we would like to do, in light of the three attributes of the standard form discussed above, is to convert the constraint set into the following "target" configuration:

$$\begin{array}{llllllllll}
z & & +? & & & & +? & = & ? & (0) \\
 & & +? & +s_1 & & & +? & = & ? & (1) \\
 & & +? & & +s_2 & & +? & = & ? & (2) \\
 & & +? & & & +s_3 & +? & = & ? & (3) \\
 & x_1 & +? & & & & +? & = & ?\,, & (4)
\end{array}$$

where the ?'s represent blanks whose entries are to be determined. Note that in this new configuration, s_1, s_2, and s_3 continue to serve, respectively, as the basic variables associated with equations (1), (2), and (3); and that the variable x_1 now replaces s_4 as the basic variable associated with equation (4).

To create this target equation system, we will employ two types of *row operations*: (i) multiplying an equation by a nonzero number, and (ii) adding one equation to another. While such row operations do create a change in the appearance of an equation system, they clearly do not have any effect on its solution.

Since the variables s_1, s_2, and s_3 are to retain their original configuration, we will focus on manipulating terms that involve x_1; and while doing so, we will aim to leave terms that involve s_1, s_2, and s_3 undisturbed. This will be done via a series of row operations, described below.

Consider equation (4) first. In the original equation system, x_1 has a coefficient of 2; whereas in the target equation system, a coefficient of 1. Hence, dividing the original equation (4) by 2, which is a row operation, will yield a new equation with the desired coefficient for x_1. Doing this leads to

$$x_1 \quad +(1/2)x_2 \qquad +(1/2)s_4 \quad = \quad 2\,.$$

We now look at the other equations one at a time. Equation (0) has a coefficient of -4 for x_1 and we wish to eliminate the term $-4x_1$ to arrive at a new equation (0). To accomplish this, we multiply the original equation (4) by 2 and add the outcome of this operation to the original equation (0). The combination of these two row operations leads to

$$z \qquad -x_2 \qquad +2s_4 \quad = \quad 8\,.$$

Next, x_1 has a coefficient of 2 in equation (1). To eliminate the term $2x_1$, we multiply the original equation (4) by -1 and add the outcome to equation (1). This yields

$$+2x_2 \quad +s_1 \qquad -s_4 \quad = \quad 2\,.$$

Similarly, multiplying the original equation (4) by 3/2 and adding the outcome to equation (2) yields

$$+(7/2)x_2 \quad +s_2 \quad +(3/2)s_4 \;=\; 9\,.$$

Finally, since x_1 does not appear in equation (3), there is no need to revise that equation. In summary, we have arrived at the following explicit (target) equation system:

$$\begin{array}{llllllcll}
z & & -x_2 & & & & +2s_4 & = & 8 & (0) \\
 & & +2x_2 & +s_1 & & & -s_4 & = & 2 & (1) \\
 & & +(7/2)x_2 & & +s_2 & & +(3/2)s_4 & = & 9 & (2) \\
 & & +2x_2 & & & +s_3 & & = & 5 & (3) \\
 & x_1 & +(1/2)x_2 & & & & +(1/2)s_4 & = & 2\,. & (4)
\end{array}$$

Notice that, indeed, the configuration of terms involving s_1, s_2, and s_3 (which are to remain in the basis) is preserved throughout these row operations. This is a consequence of the fact that these three variables do not participate in equation (4) in the original configuration.

What we have just completed is a procedure that is often referred to as *Gaussian elimination*. In this particular example, we eliminated the presence of the variable x_1 in every equation except equation (4), and we converted the coefficient of x_1 in equation (4) into a 1. This paves the way for declaring x_1 a basic variable in a new solution.

With the missing entries in the target equation system explicitly constructed, we can now associate a basic feasible solution to this configuration in the same way as we did for the original configuration. That is, we will declare x_2 and s_4 as nonbasic and x_1, s_1, s_2, and s_3 as basic. Doing this immediately yields the basic feasible solution $(x_1, x_2, s_1, s_2, s_3, s_4) = (2, 0, 2, 9, 5, 0)$. This solution corresponds to point E in the graph.

As before, the objective-function value associated with the new solution (2, 0, 2, 9, 5, 0) can be read directly from equation (0). That is, it is equal to the constant 8, located at the right-hand side of that equation.

We are now ready to carry out an optimality test for the current basic feasible solution, (2, 0, 2, 9, 5, 0). From the analysis for point A, we observed that boosting the value of each nonbasic variable leads to an adjacent corner-point, or basic feasible, solution. This observation is valid in general. In the current solution, the nonbasic variables are x_2 and s_4. We will therefore consider the effect of increasing the values of these variables from their current level 0. The analysis will be essentially the same as what we did for point A, but we will now do things in a more compact manner.

A careful examination of our analysis at point A shows that it is, in fact, not necessary to mirror the entire analysis at A to determine whether or not point E is optimal. We begin with a simple inspection of the new equation (0), and observe that the current nonbasic variables x_2 and s_4 have coefficients -1 and $+2$, respectively. This implies that an *attempt* to increase (an actually increase may not be possible, in general) the value of x_2 (while holding s_4 at 0) will result in a corresponding increase in the value of z at a rate of 1; and that the corresponding potential rate of increase in z relative to an attempt to increase s_4 (while holding x_2 at 0) is -2. Hence, the objective-function value can, potentially, be improved by increasing the value of x_2, and we cannot conclude that the current basic feasible solution is optimal. We will therefore make an attempt to boost the value of x_2. Since this means that x_2 is about to enter the basis, we shall refer to x_2 as the *entering variable.*

In general, if two or more nonbasic variables have a negative coefficient in equation (0), then we need to specify a rule for choosing the entering variable. Observe that, in such a situation, a nonbasic variable with a more negative coefficient, if chosen, will produce a greater rate of improvement in the objective-function value. For this reason, the standard Simplex method calls for selecting the nonbasic variable with the most-negative coefficient as the entering variable (although this can be construed as being myopic). If, on the other hand, the coefficient of every nonbasic variable is nonnegative in equation (0), then the basic feasible solution associated with the current equation configuration is optimal. This latter condition will be referred to as the *optimality criterion.*

Now, consider an increase in x_2 of size δ. As we increase δ, from 0, we must continuously adjust the values of the current basic variables to maintain feasibility. This means that each of equations (1)–(4) imposes an upper bound on the value of δ. More explicitly, equation (1) implies that δ cannot exceed the ratio $2/2$, where the numerator comes from the right-hand side constant and the denominator comes from the coefficient of x_2 in that equation; and similarly, equations (2), (3), and (4) imply, respectively, that δ cannot exceed the ratios $9/(7/2)$, $5/2$, and $2/(1/2)$. Since the smallest of these four ratios is 1, which comes from equation (1), the best we can do is to let $\delta = 1$. Notice that with $\delta = 1$, the value of the current basic variable for equation (1), namely s_1, is reduced from 2 to 0; and this makes s_1 nonbasic in the new solution. Since s_1 is about to leave the basis, we shall refer to s_1 as the *leaving variable.* In addition, we shall refer to the process of calculating a set of ratios to determine an upper bound for δ as the *ratio test.*

Since the upper bound at $\delta = 1$ is positive, we see that an increase in x_2 by 1 will result in a new feasible solution whose z value is greater than that of the current solution (2, 0, 2, 9, 5, 0) by a magnitude of 1 (this comes from 1×1, where the first 1 is the rate of increase in z and the second 1 is the amount of increase in x_2; see the new equation (0)). We can now conclude that the current basic feasible solution is indeed not optimal. This completes the optimality test.

At this point, we have also completed what is called an *iteration* in the Simplex method. An iteration begins with a basic feasible solution that is not yet optimal, goes through a round of Gaussian elimination to generate a new basic feasible solution, and ends with an optimality test for the new solution. The need for further iterations will be determined by the outcome of the optimality test. In our example here, the starting basic feasible solution is (0, 0, 6, 3, 5, 4), and this solution does not pass the optimality test. After going through one round of Gaussian elimination, we arrive at the new basic feasible solution (2, 0, 2, 9, 5, 0). This solution, again, fails the optimality test, and this marks the end of the first iteration. Since the new solution is not optimal, further iterations are necessary.

With x_2 entering and s_1 leaving, the new basis will be x_1, x_2, s_2, and s_3. As before, we will carry out a round of Gaussian elimination to explicitly derive this new basic feasible solution. With x_1, x_2, s_2, and s_3 as the new basis, the target configuration of the equations is:

$$\begin{array}{lllllllll} z & & & +? & & & +? & = & ? & (0) \\ & & +x_2 & +? & & & +? & = & ? & (1) \\ & & & +? & +s_2 & & +? & = & ? & (2) \\ & & & +? & & +s_3 & +? & = & ? & (3) \\ & x_1 & & +? & & & +? & = & ?\,; & (4) \end{array}$$

and our task is to convert the previous configuration into this new form.

We will begin with equation (1). In the previous configuration, x_2 has a coefficient of 2 in equation (1). Hence, we will multiply this equation by 1/2 to create a coefficient of 1. This yields

$$+x_2 \quad +(1/2)s_1 \qquad -(1/2)s_4 \quad = \quad 1$$

as the new equation (1). In general, the procedure is to always begin with the equation in the previous configuration that is targeted to contain the entering variable in the new configuration after Gaussian elimination. This equation is the one that has the smallest ratio in the ratio test.

What we will do next is to multiply equation (1) in the previous configuration by a set of numbers, called *multipliers*, and add the individual outcomes to the remaining equations. (Thus, this equation plays the role of a "template.") Each of the remaining equations will have an associated multiplier, and the value of this multiplier will be chosen so that the variable x_2, if present, is eliminated from that equation as a result of this operation.

Now, in the previous configuration, x_2 has a coefficient of -1 in equation (0). Therefore, the multiplier for this equation should be 1/2. After multiplying equation (1) by 1/2 and adding the outcome to equation (0), we obtain

$$z \qquad +(1/2)s_1 \qquad +(3/2)s_4 \quad = \quad 9\,;$$

and this will be the new equation (0).

In the previous configuration, x_2 has a coefficient of 7/2 in equation (2); therefore, the multiplier for that equation should be $-7/4$. After multiplying equation (1) by $-7/4$ and adding the outcome to equation (2), we obtain

$$-(7/4)s_1 \quad +s_2 \qquad +(13/4)s_4 \;=\; 11/2\,;$$

and this will be the new equation (2).

Finally, performing similar operations for equations (3) and (4), with multipliers -1 and $-1/4$, yields

$$\begin{array}{rrrrcl} & -s_1 & +s_3 & +s_4 & = & 3 \\ x_1 & -(1/4)s_1 & & +(3/4)s_4 & = & 3/2 \end{array}$$

respectively; and these will be the new equations (3) and (4).

Summarizing, we have arrived at the following new configuration of the equations:

$$\begin{array}{cccrrrrcll} z & & & +(1/2)s_1 & & & +(3/2)s_4 & = & 9 & (0) \\ & & +x_2 & +(1/2)s_1 & & & -(1/2)s_4 & = & 1 & (1) \\ & & & -(7/4)s_1 & +s_2 & & +(13/4)s_4 & = & 11/2 & (2) \\ & & & -s_1 & & +s_3 & +s_4 & = & 3 & (3) \\ & +x_1 & & -(1/4)s_1 & & & +(3/4)s_4 & = & 3/2\,. & (4) \end{array}$$

Here, the nonbasic variables are s_1 and s_4 and the basic variables are x_1, x_2, s_2, and s_3. Therefore, the basic feasible solution associated with this configuration is readily seen as (3/2, 1, 0, 11/2, 3, 0), which corresponds (as expected) to point D in the graph. Moreover, the objective function value associated with this solution equals 9, taken from the right-hand side of equation (0).

An inspection of equation (0) shows that the coefficients of both nonbasic variables are nonnegative, which, according to the optimality criterion, means that further improvements in z is not possible. Hence, the current basic feasible solution is optimal, and the solution of this example by the Simplex method is complete.

At this point, you should reflect carefully on the entire solution process to convince yourself that it does not explicitly rely on the availability of a graphical representation of the feasible set. That is, the solution procedure is completely algebraic. Furthermore, the applicability of this procedure does not depend on the number of decision variables.

It is also important to realize that it is not necessary for the Simplex method to visit all of the basic feasible solutions before determining which one is optimal. This remarkable feature can be attributed to the fact that the objective function is linear (see Figure LP-6). In our example, there are five basic feasible solutions, but only three out of these five are (explicitly) visited. Thus, the Simplex method, indeed, offers a significant reduction in the search effort, when compared with Procedure Search discussed in the previous section. For arbitrary linear programs, the degree of reduction in effort will be problem specific. However, empirical experience with a wide range of problems has shown that the reduction in effort is often substantial.