The Simplex Method in Tabular Form

In its original *algebraic form*, our problem is:

$$
\begin{array}{llllllllll}
\text{Maximize} & z \\
\text{Subject to:} \\
& z & -4x_1 & -3x_2 & & & & & = 0 & (0) \\
& & 2x_1 & +3x_2 & +s_1 & & & & = 6 & (1) \\
& & -3x_1 & +2x_2 & & +s_2 & & & = 3 & (2) \\
& & & 2x_2 & & & +s_3 & & = 5 & (3) \\
& & 2x_1 & +x_2 & & & & +s_4 & = 4 & (4)
\end{array}
$$

$$x_1,\, x_2,\, s_1,\, s_2,\, s_3,\, s_4 \ge 0\,.$$

Since the objective function and the nonnegativity constraints do not explicitly participate in the mechanics of the solution procedure, we only need to present the coefficients in the constraint equations. In tabular form, this problem will be represented as follows.

z	x_1	x_2	s_1	s_2	s_3	s_4	
1	−4	−3	0	0	0	0	0
0	2	3	1	0	0	0	6
0	−3	2	0	1	0	0	3
0	0	2	0	0	1	0	5
0	2	1	0	0	0	1	4

Thus, each constraint equation is translated into a row of coefficients; and the coefficients of a variable in different equations are listed in the same column, with the name of that variable specified at the top of that column as heading. In addition, to facilitate reading, we have delimited the table into several areas, e.g., (i) the z-column at the left, (ii) the coefficients in equation (0) at the top row, and (iii) the right-hand side constants of the equations at the right-most column.

The above table will be referred to as the initial Simplex tableau. To simplify statements, we will refer to the successive rows in the tableau as R_0, R_1, and so on; this numbering, of course, corresponds to that of the original equations. In addition, we will refer to the last column as the RHS column (since it comes from the right-hand-side constants in the equations).

Associated with this initial tableau, the nonbasic variables are x_1 and x_2 and the basic variables are s_1, s_2, s_3, and s_4. Therefore, the initial (or current) basic feasible solution is: $(x_1, x_2, s_1, s_2, s_3, s_4) = (0, 0, 6, 3, 5, 4)$. This solution has an objective-function value 0, which is the right-most number in R_0.

Consider R_0. Since the coefficients of x_1 and x_2 (the nonbasic variables) in that row are both negative, the current solution is not optimal. Furthermore, since the coefficient of x_1, namely -4, is more negative than that of x_2, we will select x_1 as the entering variable.

We will refer to the x_1-column as the *pivot column.* This terminology is suggested by the fact that a round of Gaussian elimination is also called a *pivot.*

To determine the maximum possible increase in x_1, we conduct a ratio test. The ratio test will involve the coefficients in the pivot column and in the RHS column. This is worked out on the right margin of the tableau, as shown below.

Basic Variable	z	x_1	x_2	s_1	s_2	s_3	s_4		Ratio Test	
	1	-4	-3	0	0	0	0	0		
s_1	0	2	3	1	0	0	0	6	$6/2 = 3$	
s_2	0	-3	2	0	1	0	0	3	–	
s_3	0	0	2	0	0	1	0	5	–	
s_4	0	2	1	0	0	0	1	4	$4/2 = 2$	$\leftarrow$ Minimum

Note that we did not compute a ratio for R_2 and R_3, since both of these two rows have a nonpositive coefficient in the pivot column (indicating that the corresponding equations (2) and (3) do not impose any bound on x_1). Since the minimum ratio appears in R_4, the basic variable currently associated with that row, s_4 (indicated at the left margin), will be the leaving variable. We will refer to R_4 as the *pivot row.*

With s_4 leaving and x_1 entering, the new basis will be x_1, s_1, s_2, and s_3. Therefore, we are now interested in constructing a new tableau that is targeted to assume the configuration specified below.

Basic Variable	z	x_1	x_2	s_1	s_2	s_3	s_4	
	1	0	?	0	0	0	?	?
s_1	0	0	?	1	0	0	?	?
s_2	0	0	?	0	1	0	?	?
s_3	0	0	?	0	0	1	?	?
x_1	0	1	?	0	0	0	?	?

As before, the ?'s in the tableau represent blanks whose entries are to be determined.

To create this target tableau, we will employ row operations. As examples, the new row 4 will be generated by multiplying R_4 by 1/2; and the new row 0 will be generated by multiplying R_4 by 2 and adding the outcome into R_0. Repeating similar operations for the other rows yields the new tableau below.

	z	x_1	x_2	s_1	s_2	s_3	s_4	
$2 \times R_4 + R_0$:	1	0	-1	0	0	0	2	8
$(-1) \times R_4 + R_1$:	0	0	2	1	0	0	-1	2
$(3/2) \times R_4 + R_2$:	0	0	7/2	0	1	0	3/2	9
$0 \times R_4 + R_3$:	0	0	2	0	0	1	0	5
$(1/2) \times R_4$:	0	1	1/2	0	0	0	1/2	2

Note that on the left margin of this tableau, we have explicitly indicated how individual new rows are derived from those in the initial tableau. For example, the operations leading to the new row 0 is listed as $2 \times R_4 + R_0$, which corresponds to the earlier description.

In this round of Gaussian elimination, or pivot, the entry 2 located at the intersection of the pivot column and the pivot row in the initial tableau plays a "pivotal role," in that it is repeated used to generate all five multipliers to R_4. We shall refer to this entry as the *pivot element.*

The basis associated with the new tableau is: x_1, s_1, s_2, and s_3. Therefore, the new basic feasible solution is: $(x_1, x_2, s_1, s_2, s_3, s_4) = (2, 0, 2, 9, 5, 0)$. This solution has an objective-function value 8. Since there is a negative coefficient in the new R_0, namely the -1 in the x_2-column, the current solution is not optimal. This completes the first iteration of the Simplex method.

Next, since x_2 is now the entering variable, the x_2-column is the new pivot column. To determine the pivot row, we again conduct a ratio test, which is shown below.

Basic Variable	z	x_1	x_2	s_1	s_2	s_3	s_4		Ratio Test	
	1	0	-1	0	0	0	2	8		
s_1	0	0	2	1	0	0	-1	2	$2/2 = 1$	$\leftarrow$ Minimum
s_2	0	0	7/2	0	1	0	3/2	9	$9/(7/2) = 18/7$	
s_3	0	0	2	0	0	1	0	5	$5/2$	
x_1	0	1	1/2	0	0	0	1/2	2	$2/(1/2) = 4$	

This shows that the new pivot row will be R_1, and the basic variable associated with that row, s_1, will be the leaving variable.

With the entry 2 (Where is it?) as the pivot element, we now go through another pivot to obtain the new tableau below.

	z	x_1	x_2	s_1	s_2	s_3	s_4	
$(1/2) \times R_1 + R_0$:	1	0	0	1/2	0	0	3/2	9
$(1/2) \times R_1$:	0	0	1	1/2	0	0	$-1/2$	1
$(-7/4) \times R_1 + R_2$:	0	0	0	$-7/4$	1	0	13/4	11/2
$(-1) \times R_1 + R_3$:	0	0	0	-1	0	1	1	3
$(-1/4) \times R_1 + R_4$:	0	1	0	$-1/4$	0	0	3/4	3/2

Here again, the operations that led to this tableau are indicated on the left margin.

The basic feasible solution associated with this new tableau is (3/2, 1, 0, 11/2, 3, 0), with a corresponding objective-function value of 9. Moreover, since the coefficients of s_1 and s_4 in the new R_0 are positive, this solution is optimal. This completes the second iteration, and the solution of this problem.