Sensitivity Analysis: An Example

Consider the linear program:

$$\begin{array}{llrrrcrl}
\text{Maximize} & z = & -5x_1 & +5x_2 & +13x_3 & & & \\
\text{Subject to:} & & & & & & & \\
& & -x_1 & +x_2 & +3x_3 & \leq & 20 & (1) \\
& & 12x_1 & +4x_2 & +10x_3 & \leq & 90 & (2)
\end{array}$$

$$x_1,\, x_2,\, x_3 \geq 0\,.$$

After introducing two slack variables s_1 and s_2 and executing the Simplex algorithm to optimality, we obtain the following final set of equations:

$$\begin{array}{lrrrrrcrl}
z & & & +2x_3 & +5s_1 & & = & 100\,, & (0) \\
& -x_1 & +x_2 & +3x_3 & +s_1 & & = & 20\,, & (1) \\
& 16x_1 & & -2x_3 & -4s_1 & +s_2 & = & 10\,. & (2)
\end{array}$$

Our task is to conduct sensitivity analysis by *independently* investigating each of a set of nine changes (detailed below) in the original problem. For each change, we will use the fundamental insight to revise the final set of equations (in tableau form) to identify a new solution and to test the new solution for feasibility and (if applicable) optimality.

We will first recast the above equation systems into the following pair of initial and final tableaus.

Initial Tableau:

Basic Variable	z	x_1	x_2	x_3	s_1	s_2	
	1	5	−5	−13	0	0	0
s_1	0	−1	1	3	1	0	20
s_2	0	12	4	10	0	1	90

Final Tableau:

Basic Variable	z	x_1	x_2	x_3	s_1	s_2	
	1	0	0	2	5	0	100
x_2	0	−1	1	3	1	0	20
s_2	0	16	0	−2	−4	1	10

The basic variables associated with this final tableau are x_2 and s_2; therefore, the current basic feasible solution is $(x_1, x_2, x_3, s_1, s_2) = (0, 20, 0, 0, 10)$, which has an objective-function value of 100.

An inspection of the initial tableau shows that the columns associated with z, s_1, and s_2 form a 3×3 identity matrix. Therefore, the $\mathbf{P}$ matrix will come from the corresponding columns in the final tableau. That is, we have

$$\mathbf{P} = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix};$$

and the final tableau equals the matrix product of this $\mathbf{P}$ and the initial tableau, i.e., $\mathbf{T}_F = \mathbf{P} \times \mathbf{T}_I$.

Our basic approach for dealing with parameter changes in the original problem is in two steps. In the first step, we will revise the final tableau by multiplying the *same* $\mathbf{P}$ to the new initial tableau; in other words, despite a revision in $\mathbf{T}_I$, we intend to follow the original sequence of pivots. After producing a revised $\mathbf{T}_F$, we will, in the second step, take the revised $\mathbf{T}_F$ as the starting point and initiate any necessary further analysis of the revised problem.

We now begin a detailed sensitivity analysis of this problem.

(a) Change the right-hand side of constraint (1) to 30.

Denote the right-hand-side constants in the original constraints as b_1 and b_2. Then, the proposed change is to revise b_1 from 20 to 30, while retaining the original value of b_2 at 90. With this change, the RHS column in the initial tableau becomes

$$\begin{bmatrix} 0 \\ 30 \\ 90 \end{bmatrix}.$$

Since the rest of the columns in the initial tableau stays the same, the only necessary revision in $\mathbf{T}_F$ will be in the RHS column. To determine this new RHS column, we multiply $\mathbf{P}$ to the above new column to obtain:

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 30 \\ 90 \end{bmatrix} = \begin{bmatrix} 150 \\ 30 \\ -30 \end{bmatrix}.$$

Since the basic variables in the final tableau are x_2 and s_2, the solution associated with the revised $\mathbf{T}_F$ is $(x_1, x_2, x_3, s_1, s_2) = (0, 30, 0, 0, -30)$. With a negative value for s_2, this (basic) solution is not feasible.

Geometrically speaking, increasing the value of b_1 from 20 to 30 means that we are relaxing the first inequality constraint. Relaxing a constraint is tantamount to enlarging the feasible set; therefore, one would expect an improved optimal objective-function value. The fact that the revised solution above is not feasible is not a contradiction to this statement. It only means that additional work is necessary to determine the new optimal solution.

What causes the infeasibility of the new solution? Recall that the original optimal solution is $(x_1, x_2, x_3, s_1, s_2) = (0, 20, 0, 0, 10)$. Since x_1, x_3, and s_1 are serving as nonbasic variables, the defining equations for this solution are: $x_1 = 0$, $x_3 = 0$, and $-x_1 + x_2 + 3x_3 = 20$. Now, imagine an attempt to increase the RHS constant of the last equation from 20 to $20 + \delta$ (say) while maintaining these three equalities. As we increase δ (from 0), we will trace out a family of solutions. That the new solution is infeasible simply means that if δ is made sufficiently large (in this case, $\delta = 10$), then this family of solutions will eventually exit the feasible set.

More formally, suppose the original RHS column is revised to

$$\begin{bmatrix} 0 \\ 20+\delta \\ 90 \end{bmatrix}; \text{ or alternatively, to } \begin{bmatrix} 0 \\ 20 \\ 90 \end{bmatrix} + \begin{bmatrix} 0 \\ \delta \\ 0 \end{bmatrix}.$$

Then, after premultiplying this new column by $\mathbf{P}$, we obtain

$$\begin{aligned}
\begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 20+\delta \\ 90 \end{bmatrix}
&= \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \times \left(\begin{bmatrix} 0 \\ 20 \\ 90 \end{bmatrix} + \begin{bmatrix} 0 \\ \delta \\ 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 20 \\ 90 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ \delta \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 100 \\ 20 \\ 10 \end{bmatrix} + \begin{bmatrix} 5\delta \\ \delta \\ -4\delta \end{bmatrix} \\
&= \begin{bmatrix} 100+5\delta \\ 20+\delta \\ 10-4\delta \end{bmatrix}.
\end{aligned}$$

Hence, with $\delta = 10$, we indeed have $s_2 = -30$, which means that the original inequality constraint $12x_1 + 4x_2 + 10x_3 \leq 90$ is violated. Moreover, this calculation also shows that in order for $10 - 4\delta$ to remain nonnegative, δ cannot exceed $5/2$. In other words, at $\delta = 5/2$, the family of solutions $(0, 20+\delta, 0, 0, 10-4\delta)$ "hits" the constraint equation $12x_1 + 4x_2 + 10x_3 = 90$; and therefore, progressing further will produce solutions that are outside the feasible set.

Interestingly, our analysis above holds even if we allow δ to assume a negative value. Such a case corresponds to a tightening of the constraint $-x_1 + x_2 + 3x_3 \leq 20$. A quick inspection of

$$\begin{bmatrix} 100+5\delta \\ 20+\delta \\ 10-4\delta \end{bmatrix}$$

shows that x_2 is reduced to 0 when δ reaches -20. It follows that in order to maintain feasibility, and hence optimality (since the optimality test is not affected by a change in the RHS column), of solutions of the form $(0,\ 20+\delta,\ 0,\ 0,\ 10-4\delta)$, the value of δ must stay within the range $[-20,\ 5/2]$.

Another important observation regarding the above calculation is that the *optimal* objective-function value will increase from 100 to $100+5\delta$, provided that δ is sufficiently small (so that we remain within the feasible set). If we interpret the value of b_1 as the availability of a resource, then this observation implies that for every additional unit of this resource, the optimal objective-function value will increase by 5. Thus, from an economics viewpoint, we will be unwilling to pay more than 5 (dollars) for an additional unit of this resource. For this reason, the value 5 is called the *shadow price* of this resource.

It is interesting to note that the shadow price of the first resource (5, in this case) can be read directly from the top entry in the second column of $\mathbf{P}$.

It is possible to derive a new optimal solution for the proposed new problem with $\delta = 10$. The standard approach for doing this is to start from the revised final tableau and apply what is called the *dual Simplex algorithm.* As this algorithm is more advanced, we will not attempt to solve this new problem to optimality.

(b) Change the right-hand side of constraint (2) to 70.

Since the original value of b_2 is 90, this is an attempt to reduce the availability of the second resource by 20. The analysis is similar to that in part (a). Again, we will write the new RHS column in the initial tableau as

$$\begin{bmatrix} 0 \\ 20 \\ 90 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \delta \end{bmatrix},$$

where δ is targeted to assume the value -20. After premultiplying this new column by $\mathbf{P}$, we obtain

$$\begin{aligned} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \times \left(\begin{bmatrix} 0 \\ 20 \\ 90 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \delta \end{bmatrix} \right) &= \begin{bmatrix} 100 \\ 20 \\ 10 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \delta \end{bmatrix} \\ &= \begin{bmatrix} 100 \\ 20 \\ 10+\delta \end{bmatrix}. \end{aligned}$$

Hence, for all δ within the range $[-10, \infty)$, solutions of the form $(0,\ 20,\ 0,\ 0,\ 10+\delta)$ will remain optimal.

With the particular choice of $\delta = -20$, we have

$$\begin{bmatrix} 100 \\ 20 \\ 10+\delta \end{bmatrix} = \begin{bmatrix} 100 \\ 20 \\ -10 \end{bmatrix}.$$

It follows that the new solution $(0,\ 20,\ 0,\ 0,\ -10)$ is infeasible. As in part (a), we will not attempt to derive a new optimal solution.

The shadow price of the second resource can be read directly from the top entry in the third column of $\mathbf{P}$. In this case, it is given by 0. That the shadow price of the second resource is equal to 0 is expected. It is a consequence of the fact that in the current optimal solution, we have $s_2 = 10$ and hence there is already an excess in the supply of the second resource. In fact, we will have an over supply as long as the availability of the second resource is no less than 80 (which corresponds to $\delta = -10$).

(c) Change b_1 and b_2 to 10 and 100, respectively.

Again, we will first consider a revision of the RHS column in $\mathbf{T}_I$ of the form:

$$\begin{bmatrix} 0 \\ 20 \\ 90 \end{bmatrix} + \begin{bmatrix} 0 \\ \delta_1 \\ \delta_2 \end{bmatrix},$$

where δ_1 and δ_2 are two independent changes. After premultiplying this new column by $\mathbf{P}$, we obtain

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \times \left(\begin{bmatrix} 0 \\ 20 \\ 90 \end{bmatrix} + \begin{bmatrix} 0 \\ \delta_1 \\ \delta_2 \end{bmatrix} \right) = \begin{bmatrix} 100+5\delta_1 \\ 20+\delta_1 \\ 10-4\delta_1+\delta_2 \end{bmatrix}.$$

With $\delta_1 = -10$ and $\delta_2 = 10$, the new RHS column in $\mathbf{T}_F$ is:

$$\begin{bmatrix} 50 \\ 10 \\ 60 \end{bmatrix}.$$

Since the new solution $(x_1,\ x_2,\ x_3,\ s_1,\ s_2) = (0,\ 10,\ 0,\ 0,\ 60)$ is feasible, it is also optimal. The new optimal objective-function value is 50.

(d) Change the coefficient of x_3 in the objective function to $c_3 = 8$ (from $c_3 = 13$).

Consider a revision in the value of c_3 by δ; that is, let $c_3 = 13 + \delta$. Then, the x_3-column in $\mathbf{T}_I$ is revised to

$$\begin{bmatrix} -13-\delta \\ 3 \\ 10 \end{bmatrix}; \text{ or alternatively, to } \begin{bmatrix} -13 \\ 3 \\ 10 \end{bmatrix} + \begin{bmatrix} -\delta \\ 0 \\ 0 \end{bmatrix}.$$

From the fundamental insight, the corresponding revision in the x_3-column in $\mathbf{T}_F$ is

$$\begin{aligned} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \times \left(\begin{bmatrix} -13 \\ 3 \\ 10 \end{bmatrix} + \begin{bmatrix} -\delta \\ 0 \\ 0 \end{bmatrix} \right) &= \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} + \begin{bmatrix} -\delta \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2-\delta \\ 3 \\ -2 \end{bmatrix}. \end{aligned}$$

Therefore, if $\delta = -5$, which corresponds to $c_3 = 8$, then the new x_3-column in $\mathbf{T}_F$ is explicitly given by

$$\begin{bmatrix} 2-\delta \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2-(-5) \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ -2 \end{bmatrix}.$$

Observe that the x_3-column is the only column in $\mathbf{T}_F$ that requires a revision, the variable x_3 is nonbasic, and the coefficient of x_3 in the revised R_0 is positive (7, that is). It follows that the original optimal solution $(x_1, x_2, x_3, s_1, s_2) = (0, 20, 0, 0, 10)$ remains optimal.

More generally, an inspection of the top entry in the new x_3-column,

$$\begin{bmatrix} 2-\delta \\ 3 \\ -2 \end{bmatrix},$$

reveals that the original optimal solution will remain optimal for all δ such that $2 - \delta \geq 0$, i.e., for all δ in the range $(-\infty, 2]$.

(e) Change c_1 to -2, a_{11} to 0, and a_{21} to 5.

This means that the x_1-column in $\mathbf{T}_I$ is revised from

$$\begin{bmatrix} 5 \\ -1 \\ 12 \end{bmatrix} \text{ to } \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}.$$

Since the corresponding new column in $\mathbf{T}_F$ is

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix},$$

where the top entry, 2, is positive, and since x_1 is nonbasic in $\mathbf{T}_F$, we see that the original optimal solution remains optimal.

(f) Change c_2 to 6, a_{12} to 2, and a_{22} to 5.

This means that the x_2-column in $\mathbf{T}_I$ is revised from

$$\begin{bmatrix} -5 \\ 1 \\ 4 \end{bmatrix} \text{ to } \begin{bmatrix} -6 \\ 2 \\ 5 \end{bmatrix}.$$

The fundamental insight implies that the corresponding new x_2-column in $\mathbf{T}_F$ is

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \times \begin{bmatrix} -6 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}.$$

The fact that this new column is no longer of the form

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

indicates that x_2 cannot serve as a basic variable in R_1. It follows that a pivot in the x_2-column is needed to restore x_2 back to the status of a basic variable. More explicitly, the revised final tableau is

Basic Variable	z	x_1	x_2	x_3	s_1	s_2	
	1	0	4	2	5	0	100
—	0	−1	2	3	1	0	20
s_2	0	16	−3	−2	−4	1	10

and we will execute a pivot with the x_2-column as the pivot column and R_1 as the pivot row. After this pivot, we obtain

Basic Variable	z	x_1	x_2	x_3	s_1	s_2	
	1	2	0	−4	3	0	60
x_2	0	−1/2	1	3/2	1/2	0	10
s_2	0	29/2	0	5/2	−5/2	1	40

.

Since x_3 now has a negative coefficient in R_0, indicating that the new solution is not optimal, the Simplex algorithm should be restarted to derive a new optimal solution (if any).

(g) Introduce a new variable x_4 with $c_4 = 10$, $a_{14} = 3$, and $a_{24} = 5$.

This means that we need to introduce the new x_4-column

$$\begin{bmatrix} -10 \\ 3 \\ 5 \end{bmatrix}$$

into the initial tableau. (The precise location of this new column is not important.) The corresponding new column in the final tableau will be

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \times \begin{bmatrix} -10 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -7 \end{bmatrix}.$$

Since this column has a positive entry at the top and since x_4 is nonbasic, the current optimal solution remains optimal. In an application, this means that there is insufficient incentive to engage in the new "activity" x_4.

(h) Introduce a new constraint $2x_1 + 3x_2 + 5x_3 \leq 50$.

After adding a new slack variable s_3, this inequality constraint becomes $2x_1+3x_2+5x_3+s_3 = 50$. Next, we incorporate this equation into the final tableau to obtain

Basic Variable	z	x_1	x_2	x_3	s_1	s_2	s_3	
	1	0	0	2	5	0	0	100
–	0	−1	1	3	1	0	0	20
s_2	0	16	0	−2	−4	1	0	10
s_3	0	2	3	5	0	0	1	50

.

Observe that x_2 participates in the new equation and, therefore, cannot serve as the basic variable for R_1. To rectify this situation, we will execute the row operation $(-3) \times R_1 + R_3$. This yields

Basic Variable	z	x_1	x_2	x_3	s_1	s_2	s_3	
	1	0	0	2	5	0	0	100
x_2	0	−1	1	3	1	0	0	20
s_2	0	16	0	−2	−4	1	0	10
s_3	0	5	0	−4	−3	0	1	−10

.

With $s_3 = -10$, the new basic solution is not feasible. We will not attempt to continue the solution of this new problem (as it is now necessary to apply the dual Simplex algorithm).

(i) Change constraint (2) to $10x_1 + 5x_2 + 10x_3 \leq 100$.

With this revision, the initial tableau becomes

Basic Variable	z	x_1	x_2	x_3	s_1	s_2	
	1	5	−5	−13	0	0	0
s_1	0	−1	1	3	1	0	20
s_2	0	10	5	10	0	1	100

After premultiplying this by $\mathbf{P}$, we obtain the revised final tableau below.

Basic Variable	z	x_1	x_2	x_3	s_1	s_2	
	1	0	0	2	5	0	100
–	0	−1	1	3	1	0	20
s_2	0	14	1	−2	−4	1	20

Observe that x_2 participates in R_2 and, therefore, cannot serve as the basic variable for R_1. To rectify this situation, we will execute the row operation $(-1) \times R_1 + R_2$. This yields

Basic Variable	z	x_1	x_2	x_3	s_1	s_2	
	1	0	0	2	5	0	100
x_2	0	−1	1	3	1	0	20
s_2	0	15	0	−5	−5	1	0

Therefore, the new optimal solution is $(x_1, x_2, x_3, s_1, s_2) = (0, 20, 0, 0, 0)$.